ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Monday, Oct. 2

1. Boundary value problems (BVPs) involving special DEs

   a. Primarily concerned with 2nd order DEs (most common in mathematical physics)
   b. Appear frequently in important partial differential equations (PDEs):
      i. Fourier equation and modified Fourier equation
      ii. Cauchy-Euler equation
      iii. Bessel equation
   c. Others (Legendre, Airy, …) appear less frequently but have important special applications

2. Solutions to Fourier and modified Fourier equations:

$$y'' + a^2 y = 0 \qquad \text{and} \qquad y'' - a^2 y = 0$$

   a. For closed boundaries (i.e., problem defined over finite range of x), recommend

$$y(x) = c_1 \cos(ax) + c_2 \sin(ax) \qquad \text{and} \qquad y(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$$

   Roots r_1 and r_2 of characteristic equation imaginary for Fourier equation and real for modified Fourier equation

   b. For open boundaries (i.e., problem defined over infinite or semi-infinite range of x), recommend

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

3. Example #1: BVP involving Fourier equation:

$$y'' + a^2 y = 0 \quad \text{with} \quad y(0) = 0,\ y(1) = 0$$

   a. Nontrivial solution is $y(x) = c_2 \sin(n\pi x)$, $n = 1, 2, 3, \ldots$
   b. Infinitely many nontrivial solutions since infinitely many integers n will work. This is an eigenvalue problem. Constants $a_n = n\pi$ are *eigenvalues*, and elementary solutions $\sin(n\pi x)$ are *eigenfunctions*.
   c. Compare to $A\mathbf{y} = \lambda\mathbf{y}$, where A is a linear operator (2nd order derivative in this case).
   d. The constant c_2 is left unspecified in this problem. However, if there had been a forcing function [i.e., $y'' + a^2 y = g(x)$], then c_2 could be uniquely specified.

(*continued on next page*)

4. Example #2: Now consider an arbitrary value for a^2 but same BCs:

$$y'' + 2.5\pi y = 0 \quad \text{with} \quad y(0) = 0,\ y(1) = 0$$

No nontrivial solutions because $y(1) = 0$ is not satisfied; $y = 0$ is still a solution.

5. Example #3: BVP involving modified Fourier equation:

$$y'' - a^2 y = 0 \quad \text{with} \quad y(0) = 0,\ y(1) = 0$$

   a. Attempt to apply

$$y(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$$

   b. No nontrivial solutions because $y(1) = 0$ is not satisfied; $y = 0$ is still a solution.

6. Cauchy-Euler equation

$$ax^2 y'' + bxy' + cy = 0$$
$$\text{special case: } x^2 y'' + xy' - \alpha^2 y = 0$$
$$\text{special special case: } x^2 y'' + xy' - y = 0$$

7. "Peel-the-onion" method applied to Cauchy-Euler equation with $a = b = c = 1$

$$x^2 y'' + xy' - y = 0 \quad \text{equiv. to} \quad \frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}(xy)\right] = 0$$

Successive integrations to arrive at solution. First integration w.r.t. x:

$$\frac{1}{x}\frac{d}{dx}(xy) = c_1 \quad \rightarrow \quad \frac{d}{dx}(xy) = c_1 x$$

Second integration w.r.t. x:

$$xy = c_1 \frac{x^2}{2} + c_2 \quad \rightarrow \quad y = c_1 \frac{x}{2} + c_2 \frac{1}{x}$$

8. Solution to 2nd order Cauchy-Euler equation with $a = b = 1$ (See Sec. 3.6 of Zill, 6th ed.)

$$x^2 y'' + xy' - \alpha^2 y = 0 \quad \text{solutions are} \quad y = \begin{cases} c_1 + c_2 \ln x, & \alpha = 0 \\ c_1 x^{-\alpha} + c_2 x^{\alpha}, & \alpha > 0 \end{cases}$$