ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Friday, Nov. 3

1. Interpretation of wave equation solution

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad 0 \le x \le L, \quad t \ge 0$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x), \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

where $v = \sqrt{\dfrac{T}{\rho}}$, where T = tension in string, ρ = mass per unit length

a. Full solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi vt}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi vt}{L}\right) \right]$$

with $A_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ and $B_n = \dfrac{2}{n\pi v}\displaystyle\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$

a. What can we learn from it?
b. Vibration modes and resonances; harmonically related
c. *Matlab* simulation
d. Standing waves vs. traveling waves for the $g(x) = 0$ (or $B_n = 0$) case. Use the identity

$$\sin(a)\cos(b) = \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(a-b)$$

to obtain

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right) = \frac{1}{2}\sin\left[\frac{n\pi}{L}(x+vt)\right] + \frac{1}{2}\sin\left[\frac{n\pi}{L}(x-vt)\right]$$

e. Not modeled in this example:
   i. Acoustic coupling between strings and nearby non-fixed objects
   ii. Energy dissipation within string and nearby objects
   iii. Scattering (reflections) from nearby objects
   iv. Presence of nearby objects, especially resonant cavities, which can affect sound perceived by listeners and can alter resonant frequencies to some degree

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2. Wave equation (1-D) problems with open boundaries (or boundaries so far away that they can be considered to be infinite in extent)

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad -\infty \le x \le \infty, \quad t \ge 0$$

   a. Special solution methods required (e.g., Green's functions for nonhomogeneous problems)
   b. One approach: See notes on D'Alembert's solution
   c. Solution has $x \pm vt$ in arguments of functions (traveling waves)

3. Wave equation in polar (cylindrical) coordinate system. Example: Vibrating circular membrane of radius c

$$a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) = \frac{\partial^2 u}{\partial t^2} \qquad \text{for} \quad 0 \le r \le c \quad \text{and} \quad t \ge 0$$

   a. ODEs after separation

$$r^2 R'' + rR' + \lambda r^2 R = 0 \quad \text{and} \quad T'' + \lambda a^2 T = 0$$

   b. Boundary conditions and initial conditions and their interpretation

$$u(c,t) = 0, \quad u(r,0) = f(r), \quad \text{and} \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = g(r)$$

   c. In the case of a drum being struck by a stick or mallet, $f(r) = 0$ and $g(r)$ is a pulse centered at $r = 0$.
   d. Special additional condition: Solution must be finite within boundary (for $r \le c$)