

Lecture Outline for Wednesday, Oct. 4

1. Other ODE classes with variable coefficients (2nd order only)

- a. Parametric Bessel equation of order
- ν
- (parameter is
- α
-)

$$x^2 y'' + xy' + (\alpha^2 x^2 - \nu^2) y = 0$$

- b. Modified parametric Bessel equation of order
- ν

$$x^2 y'' + xy' - (\alpha^2 x^2 + \nu^2) y = 0$$

- c. Legendre equation of order
- n

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

- d. Airy equation

$$y'' \pm a^2 xy = 0$$

- e. Chebyshev equation

$$(1-x^2)y'' - xy' + a^2 y = 0$$

2. Solutions to DEs with variable coefficients – the general approach:

- a. Power series solution

$$y(x) = \sum_{n=0}^{\infty} c_n (x-x_o)^n,$$

but most of the ones we will see are expanded about $x_o = 0$:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

- b. Ordinary points: Values of x at which variable coefficients are analytic
- c. Singular points: Values of x that are not ordinary points
- d. Singular points can regular or irregular
- e. Example: $\ln(x)$ is not analytic for $x \leq 0$, so the ordinary points are $x > 0$
- f. Power series solutions valid only over intervals for which they converge; they are often not valid over all of $-\infty < x < \infty$

(continued on next page)

3. Example: Parametric Bessel equation of order ν

$$x^2 y'' + xy' + (\alpha^2 x^2 - \nu^2) y = 0$$

- a. The “parameter” is α
- b. Series solution (just one of the two possible solutions) is the *Bessel function of the first kind*,

$$J_\nu(\alpha x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \nu + n)} \left(\frac{\alpha x}{2} \right)^{2n + \nu},$$

where $\Gamma(1 + \nu)$ is the gamma function (more info in Appendix II of Zill, 6th ed.)

- c. Second series solution, which is LI from J_ν if $\nu \neq$ integer:

$$J_{-\nu}(\alpha x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 - \nu + n)} \left(\frac{\alpha x}{2} \right)^{2n - \nu}$$

- d. Many practical applications with $\nu =$ integer. A more general second solution is the *Bessel function of the second kind*,

$$Y_\nu(\alpha x) = \frac{\cos(\nu x) J_\nu(\alpha x) - J_{-\nu}(\alpha x)}{\sin(\nu x)}.$$

Seems to be indeterminate for certain values of x when $\nu =$ integer (get 0/0).

However, it can be shown that Y_ν exists and is LI from J_ν even if ν is an integer.

- e. Thus, the general solution of the parametric Bessel equation can be written

$$y(x) = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$$

- f. The J_ν and Y_ν functions are well known and tabulated in old references (see web link: NIST Digital Library of Mathematical Functions), but most modern math software packages have Bessel functions built in (e.g., in *Matlab*, they are `besselj` and `bessely`).
- g. See “Meet the Bessels” by Prof. Maneval for info on Bessel functions of zero order.