ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Wednesday, Nov. 8

1. Wave equation in polar (cylindrical) coordinate system. Example: Vibrating circular membrane of radius c (continued)

$$a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial^2 u}{\partial t^2} \qquad \text{for} \quad 0\le r\le c \quad \text{and} \quad t\ge 0$$

$$u(c,t)=0\,, \quad u(r,0)=f(r)\,, \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(r); \quad u \text{ is finite everywhere}$$

   a. In the case of a drum being struck by a stick or mallet, $f(r) = 0$ and $g(r)$ is a pulse centered at $r = 0$.

   b. SOV solution and its interpretation (x_n = roots of J_0)

$$u(r,t)=\sum_{n=1}^{\infty}\left[A_n\cos(a\alpha_n t)+B_n\sin(a\alpha_n t)\right]J_0(\alpha_n r) \quad \text{with} \quad \sqrt{\lambda_n}=\alpha_n=\frac{x_n}{c}$$

   c. Inner product of Bessel functions; weighting function is $p(r) = r$

$$\left\langle J_0(\alpha_m r),J_0(\alpha_n r)\right\rangle=\int_0^c r\,J_0(\alpha_m r)J_0(\alpha_n r)\,dr$$

   d. Apply ICs to find expressions for $\{A_n\}$ and $\{B_n\}$ coefficients

$$u(r,0)=f(r)=\sum_{n=1}^{\infty}\left[A_n(1)+B_n(0)\right]J_0(\alpha_n r)$$

$$\rightarrow \quad \int_0^c r\,f(r)J_0(\alpha_m r)\,dr=\sum_{n=1}^{\infty}A_n\int_o^c r\,J_0(\alpha_n r)J_0(\alpha_m r)\,dr \quad \rightarrow \quad A_n=\frac{\left\langle f(r),J_0(\alpha_n r)\right\rangle}{\left\|J_0(\alpha_n r)\right\|^2}$$

$$\frac{\partial u}{\partial t}=\sum_{n=1}^{\infty}\left[-A_n a\alpha_n\sin(a\alpha_n t)+B_n a\alpha_n\cos(a\alpha_n t)\right]J_0(\alpha_n r)$$

$$\rightarrow \quad B_n=\frac{\left\langle g(r),J_0(\alpha_n r)\right\rangle}{a\alpha_n\left\|J_0(\alpha_n r)\right\|^2}$$

(*continued on next page*)

e. What does the result mean? How are the eigenvalues used and interpreted?

i. Vibration frequencies: $\omega_n = 2\pi f_n = a\alpha_n = \dfrac{a x_n}{c} \quad \rightarrow \quad f_n = \dfrac{a x_n}{2\pi c}$.

where $x_1 = 2.4048$, $x_2 = 5.5201$, $x_3 = 8.6537$, $x_4 = 11.7915$, etc.

ii. Frequencies are proportional to x_n, but…

$x_2 = 2.29x_1$, $x_3 = 3.59x_1$, $x_4 = 4.89x_1$, … (no harmonic relationships)

f. *Matlab* simulation

g. Are there standing waves? How do they compare to the vibrating string case?

2. Next: Numerical solution of PDEs using finite differences

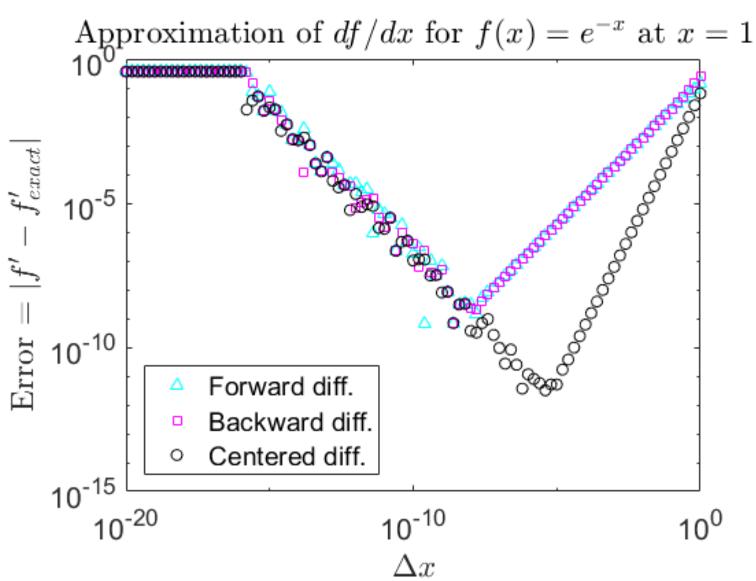
a. First derivative approximations:

i. Forward: $\dfrac{df(x_o)}{dx} \approx \dfrac{f(x_o + \Delta x) - f(x_o)}{\Delta x}$

ii. Backward: $\dfrac{df(x_o)}{dx} \approx \dfrac{f(x_o) - f(x_o - \Delta x)}{\Delta x}$

iii. Centered: $\dfrac{df(x_o)}{dx} \approx \dfrac{f(x_o + 0.5\Delta x) - f(x_o - 0.5\Delta x)}{\Delta x}$

iv. Error comparison (Δx larger on the right in graph):



b. Second derivative approximation: $\dfrac{d^2 f(x_o)}{dt^2} \approx \dfrac{f(x_o + \Delta x) - 2f(x_o) + f(x_o - \Delta x)}{\Delta x^2}$

c. Example: For $f(x) = e^x$, approximate $f'(1.2)$ using finite differences with $\Delta x = 0.1$, 0.05, and 0.01.