Chromatic Equivalence of Generalized Ladder Graphs

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Abstract

A class of graphs called generalized ladder graphs is defined. A sufficient condition for pairs of these graphs to be chromatically equivalent is proven. In addition a formula for the chromatic polynomial of a graph of this type is proven. Finally, the chromatic polynomials of special cases of these graphs are explicitly computed.
Key Words: chromatic polynomial, chromatically equivalent graphs, ladder graphs, generalized ladder graphs

Introduction and Definitions

We define topped ladders TL_n with apex a and base pair b_1 and b_2 for $n = 0, 1, 2, 3, \ldots$. We call the edge joining the base pair simply the base of the topped ladder. TL_0 is K_3 with any vertex designated as its apex and the other two vertices designated as the base pair.

We define TL_1 as TL_0 with an additional "rung" adjoined. Let TL_0 be given with apex a and base pair b_1 and b_2. Let $(r_1, r_2), (r_2, r_3),$ and (r_3, r_4) form a path of length three. Identify r_1 with b_1 in TL_0 and r_4 with b_2 in TL_0. Rename the identified vertices as r_1 and r_2. The resulting graph on seven vertices is TL_1. Designate r_2 and r_3 as the base pair of TL_1 and rename them b_1 and b_2. Repeat this process to form TL_n from TL_{n-1} and a path of length three for $n > 1$. The construction for TL_1 is shown in Figure 2.

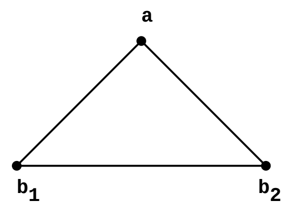


Figure 1: TL_0

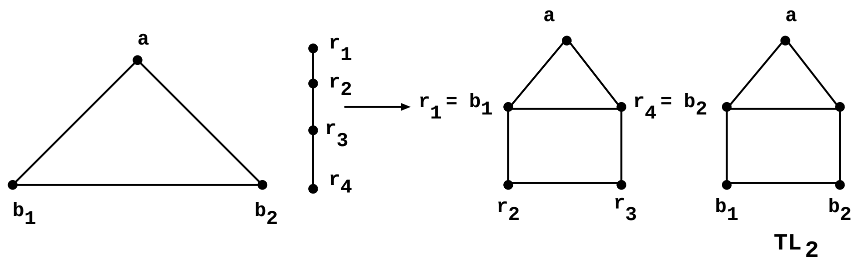


Figure 2: Constructing TL_1

We refer to TL_n simply as an $n-ladder$. TL_n for $n = 0, 1, 2, 3,$ and 4 are shown in Figure 3.

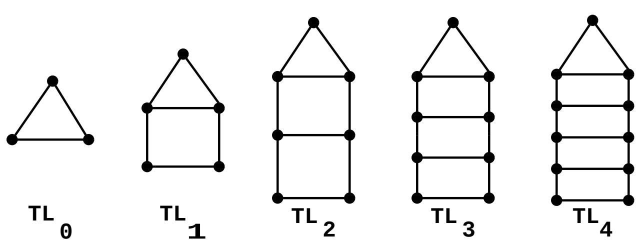


Figure 3: TL_i for $i = 0, 1, 2, 3, 4$.

We use the n-ladder graphs as building blocks to form the graphs we want to study.

Let TL_{n_1} and TL_{n_2} be n_1- and n_2-ladder graphs. Join the apex of TL_{n_1} with the apex of TL_{n_2} by an edge. Denote such a graph as $L(n_1, n_2)$. Let M be any graph such that $V(M) \cap V(L(n_1, n_2)) = \{b_1, b_2, b_3, b_4\}$, where b_1 and b_2 are the base pair for TL_{n_1} and b_3 and b_4 are the base pair for TL_{n_2}.

Let $E(M) \cap E(L(n_1, n_2)) = \{(b_1, b_2), (b_3, b_4)\}$. Define the new graph as a *generalized ladder graph* and denote it as $GL(n_1, n_2, M)$.

The results of this paper give a condition for two generalized ladder graphs to be chromatically equivalent. The chromatic polynomial of any generalized ladder graph will be computed in terms of $P(M, \lambda)$ and $P(GL(0, 0, M), \lambda)$.

In [HM] it was pointed out that if the $n - ladders$ for $n = 2, 3,$ and 4 are joined as shown in Figure 4, the resulting graphs are chromatically equivalent.

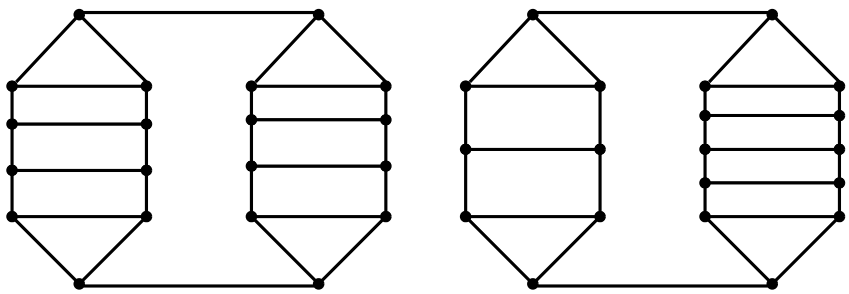


Figure 4: Chromatically Equivalent Graphs

Since these graphs are special generalized ladder pairs, we will show the fact that their chromatic polynomials are the same is no random event.

For a graph $G = (V, E)$ we use $G/(a, b)$ to denote the graph formed by deleting the edge (a, b) and then identifying the two vertices a and b as a new vertex x. Any edges incident to a and b other than (a, b) are made incident to x. This new graph is called the *contraction graph* for the edge (a, b). We use $G - (a, b)$ to denote the graph resulting from deleting the edge $a, b)$ from G; the set of vertices remains the same in this case.

The *Delete-Contract Theorem* for a graph G relative an edge (a, b) in $E(G)$ expresses the chromatic polynomial of G in terms of the chromatic polynomials of graphs formed by contracting and deleting (a, b). The theorem states

$$P(G, \lambda) = P(G - (a, b), \lambda) - P(G/(a, b), \lambda)$$

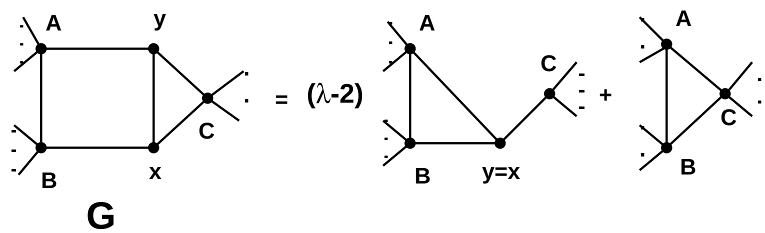
where we assume for $G - (a, b)$ that the vertex set of this graph is $V(G)$. For $G/(a, b)$ we assume the vertex set is $(V(G) - \{a, b\}) \cup \{x\}$ where $x \notin V(G)$. This is a form of Theorem 1 from [RCR. Other standard notions and results about chromatic polynomials are found in [RCR].

Fundamental Lemma

A reduction formula for a square-triangle subgraph will be the key to proving that $GL(n_1, n_2, M)$ and $GL(n_3, n_4, M)$ are chromatically equivalent when $n_1 + n_2 = n_3 + n_4$.

In the lemma below, We represent a chromatic polynomial by a diagram of the graph it comes from. More specifically, in the lemma statement we show only the part of the graph showing the square-triangle subgraph.

Lemma 1.



Proof.

$$
\begin{aligned}
P(G,\lambda) &= P(G-(y,C),\lambda) - P(G/(y,C),\lambda) \\
&= P([G-(y,C)] \cup (A,x),\lambda) \\
&\qquad + P([[G-(y,C)] \cup (A,x)]/(A,x),\lambda) \\
&\qquad - P([G/(y,C)] - (x,y=C),\lambda) \\
&\qquad + P([G/(y,C)]/(x,y=C),\lambda) \\
&= (\lambda-2)P([[G-(y,C)] \cup (A,x)] - \{(A,y),(y,x)\},\lambda) \\
&\qquad + P([[G-(y,C)] \cup (A,x)]/(A,x),\lambda) \\
&\qquad - P([G/(y,C)] - (x,y=C),\lambda) \\
&\qquad + P([G/(y,C)]/(x,y=C),\lambda)
\end{aligned}
$$

Since

$$P([[G-(y,C)] \cup (A,x)]/(A,x),\lambda) = P([G/(y,C)] - (x,y=C),\lambda),$$

we have

$$
\begin{aligned}
P(G,\lambda) = (\lambda-2)P([[G-(y,C)] \cup (A,x)] - \{(A,y),(y,x)\},\lambda) \\
+P(\{G/\{(y,C)\}\}/(x,y=C),\lambda)
\end{aligned}
$$

as required. $\bigtriangledown$

Using Lemma 1 we can prove how $P(GL(n_1, n_2, M), \lambda)$ and $P(GL(n_1 - 1, n_2 + 1, M), \lambda)$ are related.

Theorem 1. Let $n_1 > 0$. Then

$$P(GL(n_1, n_2, M), \lambda) = P(GL(n_1 - 1, n_2 + 1, M), \lambda).$$

Proof. We provide a diagrammatic proof. In Figure 5 you can see the result of applying Lemma 1 to the relevant portions of the two graphs $GL(n_1, n_2, M)$ and $GL(n_1 - 1, n_2 + 1, M)$.

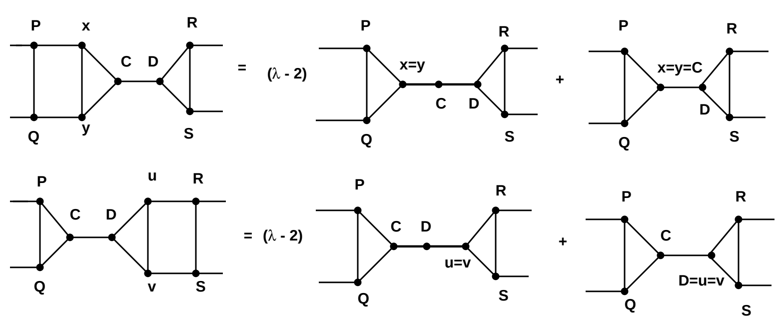


Figure 5: $P(GL(n_1, n_2, M), \lambda) = P(GL(n_1 - 1, n_2 + 1, M), \lambda)$

Since the sum of the chromatic polynomials of the graphs on the right hand sides are equal, the result follows. $\bigtriangledown$

We can now prove that graphs of the form $GL(n_1, n_2, M)$ are chromatically equivalent when n_1 and n_2 are suitably restricted.

Theorem 2. Let $GL(n_1, n_2, M)$ and $GL(n_3, n_4, M)$ be generalized ladder graphs for some graph M. If $n_1 + n_2 = n_3 + n_4$ and either (i) $n_1 > 0$ or $n_3 > 0$ or (ii) $n_1 = n_2 = 0$, then

$$P(GL(n_1, n_2, M), \lambda) = P(GL(n_3, n_4, M), \lambda).$$

Proof. If $n_1 = n_2 = 0$, the result is obvious. Without loss of generality let $n_1 > n_3$ and $n_1 > 0$. Using Theorem 1 we prove by induction on i that

$$P(GL(n_1, n_2, M), \lambda) = P(GL(n_1 - i, n_2 + i, M), \lambda).$$

Setting $i = n_1 - n_3$ proves the required result. $\bigtriangledown$

Computing the Chromial of $GL(n_1, n_2, M)$

We will find a recurrence relation for the chromial of $GL(n_1, n_2, M)$ in terms of the value of the sum $n_1 + n_2$. $GL(n_1, n_2, M)$ is shown in Figure 6.

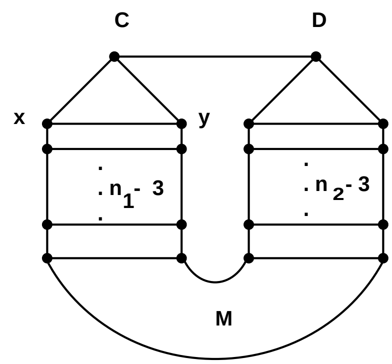


Figure 6: $GL(n_1, n_2, M)$

Theorem 3. Let $GL(n_1, n_2, M)$ be a generalized ladder graph for $n_1, n_2 \geq 0$. Then

$$\begin{aligned} P(GL(n_1, n_2, M), \lambda) &= \frac{1}{\lambda}(\lambda - 1)(\lambda - 2)^2 \\ &\quad \cdot [(\lambda^2 - 3\lambda + 3)^{n_1+n_2} - (3 - \lambda)^{n_1+n_2}] P(M, \lambda) \\ &\quad + (3 - \lambda)^{n_1+n_2} P(GL(0, 0, M), \lambda). \end{aligned}$$

Proof. We use standard techniques for computing chromatic polynomials as found in [RCR]. We need both Theorems 1 and 3 from that paper.

$$\begin{aligned} &P(GL(n_1, n_2, M), \lambda) = \\ &\qquad (\lambda - 2)P(GL(n_1, n_2, M)/(x, y), \lambda) \\ &\qquad + P([GL(n_1, n_2, M)/(x, y)]/(x = y, C), \lambda) \ \text{(by Lemma 1)} \\ &= (\lambda - 2)P([GL(n_1, n_2, M)/(x, y)] - (C, D), \lambda) \\ &\qquad - (\lambda - 2)P([GL(n_1, n_2, M)/(x, y)]/(C, D), \lambda) \\ &\qquad + P([GL(n_1, n_2, M)/(x, y)]/(x = y, C), \lambda) \\ &= (\lambda - 1)(\lambda - 2)P([[GL(n_1, n_2, M)/(x, y)] - (C, D)] - (x = y, C), \lambda) \\ &\qquad - (\lambda - 3)P([GL(n_1, n_2, M)/(x, y)]/(C, D), \lambda) \end{aligned}$$

A fuller view of the first graph on the last right hand side (see Figure 7) will make the computation of its chromial clearer.

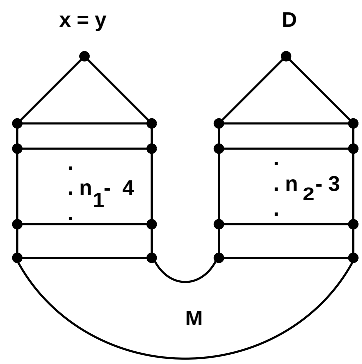


Figure 7: First Graph on the Right Hand Side

The two triangles contribute $(\lambda - 2)^2$ to the chromial of $GL(n_1, n_2, M)$. The remainder of the two ladders contribute $(\lambda^2 - 3\lambda + 3)^{n_1+n_2-1}P(M, \lambda)$ where there is a total of $n_1 + n_2 - 1$ rungs in the two ladders.

The second graph on the last right hand side has $(\lambda - 3)P(GL(n_1, n_2 - 1, M), \lambda)$ as its chromial. For clearer computations define $V_k = GL(n_1, n_2, M)$ where $n_1 + n_2 = k$ with $k \geq 0$. Let

$$\alpha = (\lambda - 1)(\lambda - 2)^3 P(M, \lambda)$$

and

$$\beta = \lambda^2 - 3\lambda + 3.$$

Then

$$\begin{aligned}
P(V_k, \lambda) &= (\lambda - 1)(\lambda - 2)^3(\lambda^2 - 3\lambda + 3)^{k-1}P(M, \lambda) - (\lambda - 3)P(V_{k-1}, \lambda) \\
&= \alpha\beta^{k-1} + (3 - \lambda)P(V_{k-1}, \lambda) \\
&= \alpha\beta^{k-1} + (3 - \lambda)\alpha\beta^{k-2} + (3 - \lambda)^2 P(V_{k-2}, \lambda) \\
&= \cdots \\
&= \alpha[\beta^{k-1} + (3 - \lambda)\beta^{k-2} + (3 - \lambda)^2\beta^{k-3} + \\
&\qquad \cdots + (3 - \lambda)^{k-1}] + (3 - \lambda)^k P(GL(0, 0, M), \lambda) \\
&= \frac{\alpha[\beta^k - (3 - \lambda)^k]}{\beta - (3 - \lambda)} + (3 - \lambda)^k P(GL(0, 0, M), \lambda)
\end{aligned}$$

$$= \frac{1}{\lambda}(\lambda-1)(\lambda-2)^2[(\lambda^2-3\lambda+3)^k-(3-\lambda)^k]P(M,\lambda) + (3-\lambda)^k P(GL(0,0,M),\lambda) \bigtriangledown$$

Application 1. Let $M = K_2$ and identify the base pairs of TL_{n_1} and TL_{n_2} to form the *rope-ladder graph* [RR] shown in Figure 8.

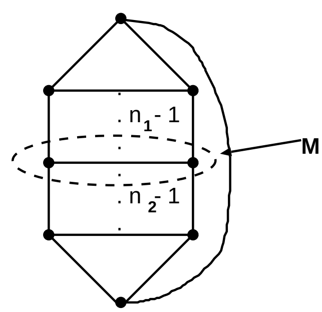


Figure 8: Rope-Ladder Graph

$$\begin{aligned} P(GL(n_1,n_2,K_2) &= (\lambda-1)^2(\lambda-2)^2(\lambda^2-3\lambda+3)^{n_1+n_2} \\ &\quad - 2(\lambda-1)(\lambda-2)(3-\lambda)^{n_1+n_2}. \end{aligned}$$

Proof. We have that

$$P(M,\lambda) = P(K_2,\lambda) = \lambda(\lambda-1)$$

and

$$P(GL(0,0,K_2),\lambda) = P(K_4,\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3).$$

Therefore,

$$\begin{aligned} P(GL(n_1,n_2,K_2),\lambda) &= \frac{1}{\lambda}(\lambda-1)(\lambda-2)^2 \\ &\quad \cdot \{(\lambda^2-3\lambda+3)^{n_1+n_2}-(3-\lambda)^{n_1+n_2}\}\lambda(\lambda-1) \\ &\quad + (3-\lambda)^{n_1+n_2}\lambda(\lambda-1)(\lambda-2)(\lambda-3) \\ &= (\lambda-1)^2(\lambda-2)^2(\lambda^2-3\lambda+3)^{n_1+n_2} \\ &\quad - 2(\lambda-1)(\lambda-2)(3-\lambda)^{n_1+n_2} \quad \bigtriangledown \end{aligned}$$

This is another proof of a computation found in [RR].

Application 2. Let M be the graph shown in Figure 9.

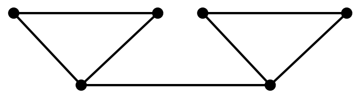


Figure 9: M

Then $GL(0,0,M)$ is as shown in Figure 10:

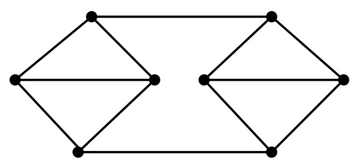


Figure 10: $GL(0,0,M)$

For the specified M, we have that:

$$P(GL(n_1,n_2,M),\lambda) = (\lambda-1)^4(\lambda-2)^4\{(\lambda^2-3\lambda+3)^{n_1+n_2}-(3-\lambda)^{n_1+n_2}\}$$
$$+(3-\lambda)^{n_1+n_2}P(GL(0,0,M).$$

Proof. Using the alternate form of Theorem 1 from [RCR], we get

$$P(M,\lambda) = \lambda(\lambda-2)^2(\lambda-1)^3.$$

$$\begin{aligned}P(GL(0,0,M),\lambda) = {} & P(GL(0,0,M)\cup(a,d),\lambda)\\ & +P(GL(0,0,M)\cup(a,d)/(a,d),\lambda).\end{aligned}$$

Using two applications of Theorem 3 from [RCR], we get

$$\begin{aligned}P(GL(0,0,M),\lambda) &= \frac{(\lambda(\lambda-1)^2(\lambda-2)^2-\lambda(\lambda-1)(\lambda-2)(\lambda-3))^2}{\lambda(\lambda-1)}\\ &+ \frac{\lambda^2(\lambda-1)^2(\lambda-2)^2(\lambda-3)^2}{\lambda}\\ &= \lambda(\lambda-1)(\lambda-2)^2(\lambda^4-7\lambda^3+19\lambda^2-25\lambda+16)\end{aligned}$$

The computation can now be completed using Theorem 3 above:

$$
\begin{aligned}
GL(n_1, n_2, M), \lambda) = {} & \\
& \frac{1}{\lambda}(\lambda-1)(\lambda-2)^2\{(\lambda^2-3\lambda+3)^{n_1+n_2} \\
& \quad - (3-\lambda)^{n_1+n_2}\}\lambda(\lambda-2)^2(\lambda-1)^3 \\
& \quad + (3-\lambda)^{n_1+n_2}P(GL(0,0,M),\lambda) \\
= {} & (\lambda-1)^4(\lambda-2)^4\{(\lambda^2-3\lambda+3)^{n_1+n_2} - (3-\lambda)^{n_1+n_2}\} \\
& \quad + (3-\lambda)^{n_1+n_2}P(GL(0,0,M),\lambda) \\
= {} & (\lambda-1)^4(\lambda-2)^4\{(\lambda^2-3\lambda+3)^{n_1+n_2} - (3-\lambda)^{n_1+n_2}\} \\
& \quad - \lambda(\lambda-1)(\lambda-2)^2(3-\lambda)^{n_1+n_2}(\lambda^4-7\lambda^3+19\lambda^2-25\lambda+16) \quad \triangledown
\end{aligned}
$$

Cycle of Ladders

The next result requires a slight generalization of the definition of a ladder. For any ladder TL_n with base pair b_1 and b_2 we define the *based ladder* TBL_n to be the graph TL_n with one new vertex b, called the base point, together with two new edges (b_1, b) and (b_2, b). We can now define a sequence of TBL_{n_i} for $1 \leq i \leq k$, called a cycle of ladders, by adding edges joining the base point of TBL_{n_1} to the apex of TBL_{n_2}, the base point of TBL_{n_2} to the apex of TBL_{n_3}, ..., and the base point of TBL_{n_k} to the apex of TBL_{n_1}. We denote a cycle of ladders as $C(n_1, n_2, \cdots n_k)$, where the n_i specify the size of the based ladders in the order in which they appear. An example of $T = C(1, 2, 3, 1)$ is shown in Figure 11.

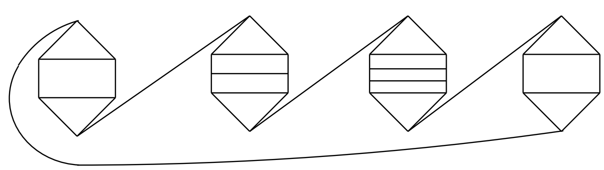


Figure 11: Cycle of Ladders

Theorem 4 leads to a characterization of those cycles of ladders that are chromatically equivalent. Finally, we can actually compute the chromatic polynomial of any cycle of ladders. We use the transformation of the cycle of ladders given in Theorem 4 to find a convenient form for computation.

Theorem 4. Let T be a cycle of ladders $C(n_1, n_2, \ldots n_k)$ composed of TBL_{n_1}, TBL_{n_2}, …, TBL_{n_k} where these graphs are connected in that order. Then

$$P(T, \lambda) = P(C(0, 0, \cdots, 0, n_1 + n_2 + \cdots + n_k), \lambda).$$

Proof. The proof is straightforward, and we leave the details to the reader. The proof involves repeated application of Theorem 2, using generalized ladders drawn from adjacent ladders in the cycle. Working around the cycle, Theorem 2 shows that if we have adjacent ladders TBL_{n_i} and $TBL_{n_{i+1}}$ where $i \in \{1, 2, \ldots, k-1\}$ the chromatic polynomial is the same for the cycle of ladders in which TBL_{n_i} and $TBL_{n_{i+1}}$ are replaced with TBL_0 and $TBL_{n_i + n_{i+1}}$, respectively. In this way all ladders but TBL_{n_k} can be transformed into copies of TBL_0 as in the statement of the theorem. $\bigtriangledown$

As an example of using this theorem, Figure 12 shows the decomposition step needed for computing the chromatic polynomial of the graph $T = C(1, 2, 3, 1)$ that was shown in Figure 11.

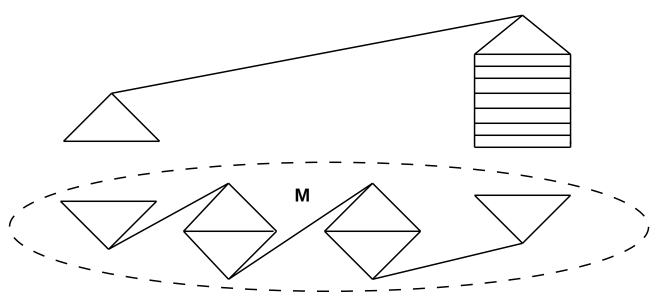


Figure 12: Decomposition for Computing the Chromatic Polynomial of a Cycle of Ladders

Using Theorem 3 of [RCR] it is easy to show that $P(M, \lambda) = \lambda(\lambda - 1)^7(\lambda - 2)^6$. The details of the rest of this computation and the general formula that can result from using Theorem 4 are left for the reader.

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