NASA/CR-1998-208462
ICASE Report No. 98-34

Graph Embeddings, Symmetric Real Matrices, and Generalized Inverses

Stephen Guattery
ICASE, Hampton, Virginia

Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, VA

Operated by Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-2199

Prepared for Langley Research Center under
Contract NAS1-19480 and NAS1-97046

August 1998

GRAPH EMBEDDINGS, SYMMETRIC REAL MATRICES, AND GENERALIZED INVERSES

STEPHEN GUATTERY*

Abstract. Graph embedding techniques for bounding eigenvalues of associated matrices have a wide range of applications. The bounds produced by these techniques are not in general tight, however, and may be off by a $\log^2 n$ factor for some graphs. Guattery and Miller showed that, by adding edge directions to the graph representation, they could construct an embedding called the current flow embedding, which embeds each edge of the guest graph as an electric current flow in the host graph. They also showed how this embedding can be used to construct matrices whose nonzero eigenvalues had a one-to-one correspondence to the reciprocals of the eigenvalues of the generalized Laplacians. For the Laplacians of graphs with zero Dirichlet boundary conditions, they showed that the current flow embedding could be used generate the inverse of the matrix. In this paper, we generalize the definition of graph embeddings to cover all symmetric matrices, and we show a way of computing a generalized current flow embedding. We prove that, for any symmetric matrix A, the generalized current flow embedding of the orthogonal projector for the column space of A into A can be used to construct the generalized inverse, or pseudoinverse, of A. We also show how these results can be extended to cover Hermitian matrices.

Key words. Laplacian matrices, graph eigenvalues and eigenvectors, graph embeddings

Subject classification. Computer Science

1. Introduction. Graph embedding techniques for bounding eigenvalues of associated matrices have a wide range of applications in problems such as bounding mixing times for Markov chains, analysis of graph separator algorithms, bounding interpolation error, and bounding spectral condition numbers of preconditioned systems. The bounds produced by these techniques are not in general tight; for certain graphs, the best lower bounds on the smallest nonzero eigenvalue of the associated generalized Laplacian matrix may be off by a $\log^2 n$ factor. Guattery and Miller showed that this is a result of the representation of the problem. By adding edge directions to the representation, they constructed an embedding called the current flow embedding, which embeds each edge of the guest graph as an electric current flow in the host graph. They showed that current flow embeddings could be used to construct matrices whose nonzero eigenvalues had a one-to-one correspondence to the reciprocals of the eigenvalues of the generalized Laplacians. For the Laplacians of graphs with zero Dirichlet boundary conditions, which are nonsinglular, they showed that the current flow embedding could be used generate the inverse of the matrix. They also discussed how to interpret embeddings without respect to edge directions in terms of those with edge directions.

In this paper, we generalize the definition of graph embeddings to cover all symmetric matrices by allowing edges with negative weights. We show a way of computing a generalized current flow embedding. We prove that, for any symmetric matrix A, the generalized current flow embedding of the orthogonal projector for the column space of A into A can be used to construct the generalized inverse, or pseudoinverse, of A. We also show how these results can be extended to cover Hermitian matrices.

*Institute for Computer Applications in Science and Engineering, Mail Stop 403, NASA Langley Research Center, Hampton, VA 23681 (smg@icase.edu). This research was supported by the National Aeronautics and Space Administration under NASA Contract Nos. NAS1-19480 and NAS1-97046 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23681.

The paper is organized as follows: Section 2 provides an overview of previous work on graph embedding bounds on eigenvalues. Section 3 describes the notation we use and covers background results used in this paper. Section 4 gives the generalization of the current flow embedding, and Section 5 shows how to use the generalized embedding techniques to construct the inverse or pseudoinverse of any symmetric real matrix.

2. Previous Work. The relationship between graph embeddings and matrix representations has been the subject of much interesting research. A large proportion of this work has been aimed at bounding the second largest eigenvalues of time-reversible Markov chains in order to bound the mixing time for random walks. The use of clique embeddings to bound eigenvalues arose in the analysis of mixing times for Markov chains by Jerrum and Sinclair [14] [17]. Further work in this direction was done by Diaconis and Strook [7] and by Sinclair [16]. Kahale [15] generalized this work in terms of methods assigning lengths to the graph edges, and showed that the best bound over all edge length assignments is the largest eigenvalue of the matrix $\Gamma^T\Gamma$, where Γ is a matrix representing the path embedding (unpublished work by Fill and Sokal in these directions is also cited in [15]). He also gave a semidefinite programming formulation for a model allowing fractional paths, and showed that the bound is off by at most a factor of $\log^2 n$. He showed this gap is tight; he also noted that the results can be applied to Laplacians with suitable modifications.

Guattery, Leighton, and Miller [12] presented a lower bound technique for λ_2 of a Laplacian. It assigns priorities to paths in the embedding, and uses these to compute congestions of edges in the the original graph with respect to the embedding. Summing the congestions along the edges in a path gives the path congestion; the lower bound is a function of the reciprocal of the maximum path congestion taken over all paths. For the clique case, they showed that this method is the dual of the method presented in [15]; the best lower bounds produced by these methods are the same. They also showed how to apply their method in the Dirichlet boundary case by using star embeddings. In the clique case, they showed that using uniform priorities for any tree T gives a lower bound that is within a factor proportional to the logarithm of the diameter of the tree.

Gremban and Miller [10] have shown how to use embeddings to generate support numbers, which also are bounds on the largest and smallest generalized eigenvalues (and hence the spectral condition number) of preconditioned linear systems involving a generalized definition of Laplacians. They also defined the support tree preconditioner, and used the support number bounds to prove properties about the quality of these preconditioners. Gremban, Miller, and Zagha have evaluated the performance of these techniques [9].

The construction of the embedding we use below is related to the effective resistances between pairs of vertices in the graph when the graph is viewed as a network of unit conductances. Chandra et al [6] have defined the maximum such value taken over all distinct pairs as the electrical resistance of a graph, and have used that quantity to bound the commute and cover times of random walks on the graph.

Independently, papers by Guattery [11] and Bern, Gilbert, Hendrickson, Nguyen, and Toledo [2] have given extensions to matrices that are not diagonally dominant.

3. Terminology, Notation, and Background Results. We assume that the reader is familiar with the basic definitions of graph theory (in particular, for undirected graphs), and with the basic definitions and results of matrix theory. A graph consists of a set of vertices V and a set of edges E. We use n to denote $|V|$ and m to denote $|E|$.

3.1. Matrices and Matrix Notation. We use capital letters to represent matrices and bold lowercase letters for vectors. For a matrix A, a_{ij} or $[A]_{ij}$ represents the element in row i and column j; for the vector $\mathbf{x}$, x_i or $[\mathbf{x}]_i$ represents the i^{th} entry. The notation $\mathbf{x} = \mathbf{0}$ indicates that all entries of the vector $\mathbf{x}$

are zero; **1** indicates the vector that has 1 for every entry. Unless specifically noted otherwise, we index the eigenvalues of an $n \times n$ matrix in non-decreasing order: λ_1 represents the smallest eigenvalue, and λ_n the largest. We use the notation $\lambda_i(A)$ (respectively $\lambda_i(G)$) to indicate the i^{th} eigenvalue of matrix A (respectively of the Laplacian of graph G) if there is any ambiguity about which matrix (respectively graph) the eigenvalue belongs to. The notation $\mathbf{u}_i$ represents the eigenvector corresponding to λ_i.

I_k denotes the $k \times k$ identity matrix.

In this paper, the term "symmetric" implies that the matrix involved is real unless otherwise explicitly stated.

The **range** or **column space** of a matrix is the span of its columns. We denote the range of matrix A by $R(A)$.

Let M be a vector subspace. The **orthogonal projector** for M is the matrix denoted by P_M such that for any $\mathbf{x} \in M$, $P\mathbf{x} = \mathbf{x}$, and for any $\mathbf{x} \perp M$, $P\mathbf{x} = \mathbf{0}$. Orthogonal projectors are symmetric matrices when M is a vector space on the real numbers. The **generalized inverse** or **pseudoinverse** of a matrix A is the unique matrix A^+ such that $AA^+ = P_{R(A)}$ and $A^+A = P_{R(A^+)}$ (this is Definition 1.1.2 from [5]).

3.2. The Laplacian Matrix Representation of a Graph. A common matrix representation of unweighted graphs is the **Laplacian**. Let D be the matrix with $d_{ii} = \text{degree}(v_i)$ for $v_i \in V(G)$, and all off-diagonal entries zero. Let A be the adjacency matrix for G ($[A]_{ij} = 1$ iff $(v_i, v_j) \in E(G)$, 0 otherwise). Then the Laplacian of G is the matrix $L = D - A$. Fiedler extended the notion of the Laplacian to graphs with positive edge weights [8]; he referred to this representation as the **generalized Laplacian**. Let w_{ij} be the (positive) weight of edge (i, j) in graph G. Then the entries of the generalized Laplacian L of G are defined as follows: l_{ii} is the sum of the weights of the edges incident to vertex v_i; for $i \neq j$ and $(v_i, v_j) \in E(G)$, $l_{ij} = -w_{ij}$, and $l_{ij} = 0$ otherwise. By extending this notion to negative weights as well, any symmetric matrix with zero row sums can represent a weighted graph and vice versa.

If a matrix has a nonzero row sum, the surplus (or deficit) of the diagonal entry can be intrepreted in terms of a **zero Dirichlet boundary condition**. If r_i is the sum of the elements in the i^{th} row and d_i is the diagonal entry in row i, then the difference $d_i - r_i$ can be viewed as the weight of an edge from vertex i to a boundary vertex not explicitly represented in the matrix. A matrix representing a graph with a zero Dirichlet boundary condition can be constructed by deleting the rows and columns corresponding to the boundary vertices from the matrix representing the full graph.

It is well known (see e.g. [4, 3]) the the Laplacian of a graph G can be expressed as the product B^TB of G's **edge-vertex incidence matrix** B (denoted $B(G)$ if the corresponding graph is not clear from context). B is constructed as follows: Arbitrarily direct the edges of G, and index them. B is an $m \times n$ matrix, where row i has a -1 in the column corresponding to the vertex at its tail, a 1 in the column corresponding to the vertex at its head, and zeros in all other columns. Generalized Laplacians and matrices representing weighted graphs can also be expressed as a product involving the edge-vertex incidence matrix B. However, we must introduce an edge weight matrix W (also referred to as the **conductance matrix** when all weights are positive). W is a diagonal matrix with entry $[W]_{ii}$ equal to the weight of edge i. It is easy to check that B^TWB is equal to the matrix representing the corresponding graph.

The definition of the edge-vertex incidence matrix can be extended to the zero Dirichlet boundary condition case. In this case, the boundary vertex is not explicitly represented in the matrix. Thus, only one end of the edge is represented in the edge-vertex incidence matrix; by convention, we direct all edges toward the boundary, so the explicit entry will be a -1 for the tail of the edge. The weight of the corresponding edge is included in the weight matrix W as for any other edge. The matrix of the graph is again B^TWB.

3.3. Graph Subspaces with Respect to the Laplacian. Let A be the generalized Laplacian of a weighted graph G on n vertices and m edges. The set of vectors of length n is the **vertex space**; each entry of the vector $\mathbf{x}$ assigns the value x_i to corresponding vertex i (such an assignment of values is a **valuation**). Likewise, the set of vectors of length m is the **edge space**. Since we will use an electrical analogy below, we sometimes refer to the values assigned to vertices (or **nodes** in electrical terminology) as **potentials**.

The edge space of the graph can be partitioned into two orthogonal subspaces (see e.g. [1, 4]). The first is the **cycle space**, the subspace spanned by vectors representing cycles in G. A cycle C is represented by a vector as follows: The edges of C have a natural direction, where each edge is directed toward its successor in the cyclic order. For each edge e in C, the vector $\mathbf{v}_C$ has entry 1 if the direction of e with respect to C and the direction of e used in defining the edge-vertex incidence matrix B are the same, and -1 if they are opposite. The entries for edges not in C are 0.

The cycle space is the orthogonal complement of the **cocycle** or **cut space**, the subspace spanned by vectors representing cuts. Let $(S, \overline{S})$ represent a partition of the vertices of a graph G into two sets. The set of edges with one end in S and the other in $\overline{S}$ is called a **cut**. Cuts are represented as vectors as follows: for each edge in the cut, if the edge has its tail in S (i.e., the edge is directed from S to $\overline{S}$), the vector entry for that edge is 1. If the edge has its head in S (i.e., the edge is directed from $\overline{S}$ to S), the vector entry for that edge is -1. Entries for edges not in the cut are 0.

Some simple but useful results about the relation of the cycle space and cut spaces to the edge-vertex incidence matrix B are stated in the following lemma (the proofs are easy and left to the reader).

LEMMA 3.1. *Let G be a connected graph with edge-vertex incidence matrix B. Then*

- *The span of the columns of B is the cut space of G, and any $n-1$ columns of B form a basis for the cut space.*
- *The cycle space of G is the null space of B^T.*

3.4. Laplacians and Electrical Circuits. Graphs with positive edge weights can be viewed as resistive electrical circuits, with the edge weights representing conductances between the vertices or nodes. Guattery and Miller [13] used this idea in defining the current flow embedding of one graph into another, in which the paths embedding an edge into a graph correspond to the current flows on the edges of the graph when a unit current is injected at one endpoint of the edge and is removed at the other. Calculations of current flows can be defined in terms of three electrical laws, which we represent in matrix terms.

The first is **Ohm's law**, which connects potential differences and currents. Consider any valuation of the vertices as a vector of potentials or **voltages**. The **potential difference** across any edge is the value at the tail of the edge minus the value at the head. Thus, given a vector $\mathbf{v}$ of node voltages, $-B\mathbf{v}$ gives the vector of potential differences, which is in the edge space. Ohm's law says that the current on an edge is the potential difference times the conductance. Thus, the vector $\mathbf{i}$ of edge currents can be defined as $\mathbf{i} = -WB\mathbf{v}$, where W is the conductance matrix as discussed in Section 3.3. If external voltage sources are included, we represent them as a vector $\mathbf{v}_{ext}$, and Ohm's law becomes $\mathbf{i} = W(\mathbf{v}_{ext} - B\mathbf{v})$.

The second is **Kirchoff's voltage law (KVL)**, which states that the potential drops around any cycle in the graph sum to zero. This requires that for any cycle $\mathbf{c}$, $\mathbf{c}^T B\mathbf{v} = 0$. The second point in Lemma 3.1 shows this holds in the matrix representation.

The third law is **Kirchoff's current law (KCL)**, which states that the net current flow at any vertex is zero; that is, the sum of the currents out over all edges incident to a vertex is equal to the current injected at that vertex. Note that multiplying a vector of currents (in the edge space) by B^T sums the currents into the vertices. Therefore, if $\mathbf{i}_{ext}$ is a vector of external currents extracted from the circuit, KCL says that the

resulting currents in the circuit (represented by the vector $\mathbf{i}$) must obey

$$B^T\mathbf{i} = \mathbf{i}_{ext}. \tag{3.1}$$

(If the entries of $\mathbf{i}_{ext}$ do not sum to zero, the surplus current must be extracted at the boundary. This requires the zero Dirichlet boundary case; note that for a generalized Laplacian without such a boundary condition, all column sums of B^T are zero, and such a vector $\mathbf{i}_{ext}$ will not lie in the column space.)

Consider the following matrix expressed in block matrix form:

$$\begin{bmatrix} W^{-1} & B \\ B^T & 0 \end{bmatrix}$$

It is well known (see, e.g., Section 2.3 of [18]) that if the entries of W are positive, this matrix can be used to calculate the voltages and currents in an electrical circuit. In that case, W is the conductance matrix of the graph, and W^{-1} contains the resistances of the edges. To make the matrix nonsingular, a single node in the circuit is set to ground. A right-hand side is constructed from vectors $\mathbf{v}_{ext}$ and $\mathbf{i}_{ext}$ representing the external voltage and current sources. The former are represented as voltage differences applied across edges (in series with the edge resistance); the latter are represented as directed flows into and out of nodes in the circuit. Solving the following system gives the vectors $\mathbf{i}$ (the edge currents) and $\mathbf{v}$ (the voltages at the nodes):

$$\begin{bmatrix} W^{-1} & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{ext} \\ \mathbf{i}_{ext} \end{bmatrix}.$$

It is easy to see that, given the external sources, Ohm's law and Kirchoff's laws are satisfied by the solution of this system [18]. Ohm's law and Kirchoff's current law are explicit in the structure of the matrix problem. Kirchoff's voltage law can be stated as follows: For any cycle $\mathbf{c}$ in the cycle space of the circuit, $\mathbf{c}^T B\mathbf{v} = 0$. Since the cycle space is the null space of B^T (see Section 3.3), this clearly holds.

3.5. Graph Embeddings. Let G and H be connected graphs such that the vertex set of H is a subset of the vertex set of G. An **embedding** of H into G is a collection Γ of path subgraphs of G such that for each edge $(v_i, v_j) \in E(H)$, the embedding contains a path γ_{ij} from v_i to v_j in G. For full generality, we will allow fractional paths in our embeddings: i.e., an edge $(v_i, v_j) \in E(H)$ can be associated with a finite collection of paths from v_i to v_j in G; each such path has a positive fractional weight associated with it such that the weights add up to 1. If a path γ includes edge e, we say that γ is incident to e. Embeddings under this definition can be considered as unit flows from v_i to v_j, and can be represented in matrix form.

In [13], we defined a *current flow embedding* of an unweighted graph H into a weighted graph G. G is treated as a network of conductances (the conductance of each edge is its (positive) weight), and each edge (u, v) in H is embedded as the current flow in G when a unit current is injected at u and extracted at v. We coupled this definition with a matrix representation of the embedding that included edge directions: The entry for edge (v_i, v_j) in G in embedding an edge from H includes not only a magnitude indicating the fractional weight carried, but also a sign indicating whether the direction of the flow corresponds to the direction of (v_i, v_j) or not. Directions for the edges in G are chosen arbitrarily as for the edge-vertex incidence matrix; positive signs indicate agreement of the flow with the direction of the underlying edge.

4. A Generalization to Real Symmetric Matrices. In this section we generalize the idea of current flow embeddings to all real symmetric matrices. The physical analogy with resistive circuits breaks down because our interpretation of symmetric matrices as graphs will include negative edge weights in the general case. These would be "negative resistors" in the resistive model, and hence would have no physical analog.

However, we show that it is possible to construct an analog of Γ_{cf} such that the conditions of Ohm's and Kirchoff's laws (generalized for negative resistances) still hold.

We propose extending the definition of current flow embeddings in two ways: First, we want to allow for the possibility of negative weights in G. This requires dropping the electrical interpretation, and presenting a way of calculating the embedding. Second, we want to relax the condition that only unit currents are embedded. In particular, it will be useful in some applications to embed multiple weighted edges, which can be represented in the original model as multiple currents injected and extracted at various points.

As noted in section 3.2, we can express any symmetric matrix A in terms of an edge-vertex incidence matrix B and a real diagonal weight matrix W such that $A = B^T W B$. Since edges by definition have nonzero weights, W is invertible. We assume that the graph of A has n non-boundary vertices and m edges (including edges to the boundary). Now recall the following matrix presented in Section 3.4:

$$M = \begin{bmatrix} W^{-1} & B \\ B^T & 0 \end{bmatrix}.$$

Such a matrix exists for any symmetric matrix.

Let $\mathbf{b} = [0,\ \mathbf{i}_{ext}]^T$, where 0 is the zero vector of length m and $\mathbf{i}_{ext}$ is a vector of length n. Assume that there exists a solution $\mathbf{x}$ to the system $M\mathbf{x} = \mathbf{b}$, and write the solution as $\mathbf{x} = [\mathbf{i},\ \mathbf{v}]^T$, where $\mathbf{i}$ and $\mathbf{v}$ are an m- and an n-vector respectively:

$$\begin{bmatrix} W^{-1} & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{i}_{ext} \end{bmatrix},$$

If the solution is not unique, set $\mathbf{x}$ to the unique solution that is orthogonal to the null space of M. The *generalized current flow embedding* of $\mathbf{i}_{ext}$ is defined to be $\mathbf{i}^T$ from this solution. More generally, we will want to consider the case where the right-hand side of the system has more than one vector; in that case we write the system as

$$\begin{bmatrix} W^{-1} & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{cf}^T \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ C \end{bmatrix}. \tag{4.1}$$

If this system has a solution, we define the matrix Γ_{cf} to be the generalized current flow embedding of C into A (again, if the solution is not unique, we specify the unique solution orthogonal to the null space of M).

This terminology is drawn from the study of electrical circuits, which are modeled by the system in (4.1) when W has positive diagonal entries. In that case, the rows of Γ_{cf} are the current flows in the circuit corresponding to A when the currents represented by the columns of C are injected. C is analogous to the transpose of $B(H)$ in the case where H is embedded into G, except that the columns of C can represent a superposition of weighted currents injected and extracted rather than individual unit currents.

Even though negative resistances won't permit a physical analogy, the matrix equations that express electrical laws still hold, so we can think of them as generalized laws:

- Rewriting the second row of the product in (4.1) gives $\Gamma_{cf} B = C^T$. This is equivalent to Kirchoff's Current Law in the electrical case.
- Potential differences across the edges obey an analog of Kirchoff's Voltage Law: For any cycle $\mathbf{c}$ in the cycle space of G,

  $$\mathbf{c}^T B \mathbf{v} = 0.$$

  This follows because $\mathbf{c}^T B$ is always $\mathbf{0}$ (see Section 3.3).

- For (4.1), the generalized version of Ohm's Law (see Section 3.4) can be rewritten as $\Gamma_{cf} = WB\mathbf{v}$.

So far we have assumed that a solution to $M\mathbf{x} = \mathbf{b}$ exists for $\mathbf{b}$ specified as above. We now show the conditions under which a solution exists.

It is useful to consider the system after applying Gaussian elimination to the first m rows (i.e., using the entries of W^{-1} as pivots). The partially reduced matrix can be represented as follows:

$$\begin{bmatrix} I_m & 0 \\ -B^T W & I_n \end{bmatrix} \begin{bmatrix} W^{-1} & B \\ B^T & 0 \end{bmatrix} = \begin{bmatrix} W^{-1} & B \\ 0 & -A \end{bmatrix}$$

Note that the Schur complement from the elimination on the first m rows is $-A$. Since W^{-1} is diagonal, and the diagonal entries are nonzero, we immediately have the following lemma:

LEMMA 4.1. $\operatorname{rank}(M) = m + \operatorname{rank}(A)$.

LEMMA 4.2. *For an $n \times k$ matrix C, the system in (4.1) has a solution whenever the columns of C are in the column space of A.*

Proof. If the original matrix A is nonsingular, then a solution clearly exists. If not, note that the elimination does not affect the right-hand side of the system. Thus the reduced system can be written as follows:

$$\begin{bmatrix} W^{-1} & B \\ 0 & -A \end{bmatrix} \begin{bmatrix} \Gamma_{cf}^T \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ C \end{bmatrix}.$$

Because W^{-1} is nonsingular, the system has a solution whenever $-AV = C$ has a solution. This will be the case whenever C is in the column space of A, which proves the lemma. □

The following lemmas about the null space of M will be useful later.

Let $\mathcal{N}(M)$ be the null space of M, where M is defined with respect to the matrix A. Let $\mathbf{x} \in \mathcal{N}(M)$; $\mathbf{x}$ is a vector of length $m + n$. We will refer to the subvector containing the first m entries as $\mathbf{x}_\Gamma$, and to the subvector containing the last n entries as $\mathbf{x}_V$.

LEMMA 4.3. *For any pair of (not necessarily distinct) vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{N}(M)$, $\mathbf{x}_\Gamma^T W^{-1} \mathbf{y}_\Gamma = 0$.*

Proof. From the structure of M and the fact that $M\mathbf{x} = \mathbf{0}$, we have $W^{-1}\mathbf{y}_\Gamma = -B\mathbf{y}_V$. From the structure of M we also get that $B^T\mathbf{x}_\Gamma = \mathbf{0}$ (note that this implies that $\mathbf{x}_\Gamma$ is in the null space of B^T, which is the cycle space of $G(A)$), which implies that $\mathbf{x}_\Gamma^T B = \mathbf{0}$. Combining these gives the desired result: $\mathbf{x}_\Gamma^T W^{-1} \mathbf{y}_\Gamma = -\mathbf{x}_\Gamma^T B \mathbf{y}_V = 0$. □

LEMMA 4.4. *For any $\mathbf{x} \in \mathcal{N}(M)$, $\Gamma_{cf} W^{-1} \mathbf{x}_\Gamma = \mathbf{0}$.*

Proof. As in the previous lemma, we have $B^T\mathbf{x}_\Gamma = \mathbf{0}$, which implies that $\mathbf{x}_\Gamma^T B = \mathbf{0}$. The calculation of Γ_{cf} in (4.1) gives $W^{-1}\Gamma_{cf}^T = -BV$. Multiplying on the left by $\mathbf{x}_\Gamma^T$ and combining gives the desired result:

$$\mathbf{x}_\Gamma^T W^{-1} \Gamma_{cf}^T = -\mathbf{x}_\Gamma^T BV = \mathbf{0}.$$

□

5. Inverses and Generalized Inverses. In [13], Guattery and Miller showed that, for any generalized Laplacian matrix L with a zero Dirichlet boundary condition (i.e., a surplus on the diagonal), the current flow embedding could be used to construct the inverse L^{-1}. In particular, let Γ_{cf} be the current flow embedding derived from embedding a unit current flow from each internal vertex to ground, and let W be the conductance matrix for L (we assume that Γ_{cf} and W have been ordered consistently). Then Theorem 6.4 from [13] shows that $L^{-1} = \Gamma_{cf} W^{-1} \Gamma_{cf}^T$.

We now generalize this result to show that, for any symmetric matrix A, we can find a generalized current flow embedding Γ_{cf} from which we can construct the generalized inverse A^+. We will also show that $\Gamma_{cf} W^{-1} \Gamma_{cf}^T = A^+$. The proofs in the previous paper do not work, however, because circuit models of symmetric matrices may have "negative resistors:" some of the matrices involved become complex symmetric (but not Hermitian), and such matrices do not work in these arguments.

Let A be a symmetric matrix whose graph has n internal vertices and m edges (including edges from internal vertices to the boundary). As noted previously, we can write A as $A = B^T W B$. The matrices W and B are defined as previously, with any diagonal surpluses handled as they were for the zero Dirichlet boundary case. Note that W may now have negative entries. Now consider the following system:

$$\begin{bmatrix} W^{-1} & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{cf}^T \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ P_{R(A)} \end{bmatrix} \tag{5.1}$$

Since $P_{R(A)}$ is in the column space of A, a solution $[\Gamma_{cf}, V^T]^T$ exists by Lemma 4.2. If multiple solutions exist, we will use the unique one that is orthogonal to the null space in accordance with the definition of the generalized current flow embedding.

We now consider the properties of the components of the the solution, Γ_{cf} and V. The structure of this system is the same as for resistive circuits, as discussed in Section 4. Note in particular that for the right-hand side we are using, $B^T \Gamma_{cf}^T = P_{R(A)}$, which implies $\Gamma_{cf} B = P_{R(A)}$. If there are negative entries, the rows of Γ_{cf} are not flows, though they do meet the conditions expressed by the generalized versions of the electrical laws. However, the following relationship between Γ_{cf}, W^{-1}, and the inverse or pseudoinverse of A is the same whether there are negative weights in $G(A)$ or not:

THEOREM 5.1. $A^+ = -\Gamma_{cf} W^{-1} \Gamma_{cf}^T$

Proof. By Definition 1.1.2 in [5], it is sufficient to show that

$$-A \cdot \Gamma_{cf} W^{-1} \Gamma_{cf}^T = P_{R(A)} \tag{5.2}$$

and

$$-\Gamma_{cf} W^{-1} \Gamma_{cf}^T \cdot A = P_{R(\Gamma_{cf} W^{-1} \Gamma_{cf}^T)} \tag{5.3}$$

To see that (5.2) holds, we use the following equalities:

$$W^{-1} \Gamma_{cf}^T = BV, \tag{5.4}$$

which follows from the first row of the original system;

$$\Gamma_{cf} B = P_{R(A)}, \tag{5.5}$$

which follows from the second row of the original system; and

$$-AV = P_{R(A)}, \tag{5.6}$$

which follows from the second row of the reduced system. We also use the fact that $A P_{R(A)} = A$, which follows from the definition of the orthogonal projector ($P_{R(A)} A = A$), the symmetry of A, and the symmetry of orthogonal projectors. Starting with $-A \cdot \Gamma_{cf} W^{-1} \Gamma_{cf}^T$, we can apply these equalities as follows:

$$-A \cdot \Gamma_{cf} W^{-1} \Gamma_{cf}^T = -A \Gamma_{cf} B V = -A P_{R(A)} V = -AV = P_{R(A)}.$$

This proves (5.2).

To demonstrate that (5.3) holds, we will show that $P_{R(A)} = P_{R(\Gamma_{cf} W^{-1} \Gamma_{cf}^T)}$. This will prove the theorem, since

$$\Gamma_{cf} W^{-1} \Gamma_{cf}^T \cdot A = \left(A \cdot \Gamma_{cf} W^{-1} \Gamma_{cf}^T \right)^T = P_{R(A)}^T = P_{R(A)}. \tag{5.7}$$

To show that $P_{R(A)} = P_{R(\Gamma_{cf} W^{-1} \Gamma_{cf}^T)}$, it is sufficient to show that $\Gamma_{cf} W^{-1} \Gamma_{cf}^T$ and A have the same column spaces. Note that (5.7) implies that $R(A) \subseteq R(\Gamma_{cf} W^{-1} \Gamma_{cf}^T)$. We can apply (5.4) and (5.5) to show that

$$\Gamma_{cf} W^{-1} \Gamma_{cf}^T = \Gamma_{cf} B V = P_{R(A)} V.$$

This shows that $R(\Gamma_{cf} W^{-1} \Gamma_{cf}^T) \subseteq R(A)$, which proves the claim that $P_{R(A)} = P_{R(\Gamma_{cf} W^{-1} \Gamma_{cf}^T)}$, and thus the theorem. □

We can use Lemmas 4.3 and 4.4 to prove the following theorem. For symmetric matrix $A = B^T W B$ and $M = \begin{bmatrix} W^{-1} & B \\ B^T & 0 \end{bmatrix}$, let $[\Gamma, V^T]^T$ be any solution of

$$\begin{bmatrix} W^{-1} & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \Gamma^T \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ P_{R(A)} \end{bmatrix}.$$

THEOREM 5.2. $A^+ = -\Gamma W^{-1} \Gamma^T$.

Proof. By the definition of the current flow embedding, $\Gamma^T = \Gamma_{cf}^T + X_\Gamma$, where each column of X_Γ is a subvector consisting of the first m entries of a vector $\mathbf{x}$ in the null space of M. Then

$$\Gamma W^{-1} \Gamma^T = X_\Gamma^T W^{-1} X_\Gamma + X_\Gamma^T W^{-1} \Gamma_{cf}^T + \Gamma_{cf} W^{-1} X_\Gamma + \Gamma_{cf} W^{-1} \Gamma_{cf}^T = \Gamma_{cf} W^{-1} \Gamma_{cf}^T.$$

The first equality follows by linearity. The second follows by considering the entries of the matrices in the first three terms of the middle expression entry by entry: all entries for the first term are zero by Lemma 4.3, and all entries for the second and third terms (which are transposes of each other) are zero by Lemma 4.4. □

This theorem can be extended to the Hermitian case by the following technique, which is based on a suggestion made by John Lafferty. Note that any $n \times n$ Hermitian matrix A can be written as $\mathrm{Re}(A) + i\mathrm{Im}(A)$, where $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ are real matrices, $\mathrm{Re}(A)$ is symmetric, and $\mathrm{Im}(A)$ is skew symmetric. Let

$$H = \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}.$$

Define the unitary matrix T as follows:

$$T = \begin{bmatrix} I_n & i\, I_n \\ i\, I_n & I_n \end{bmatrix}.$$

Applying T, we can compute the similarity transform

$$THT^* = \begin{bmatrix} \mathrm{Re}(A) & \mathrm{Im}(A) \\ -\mathrm{Im}(A) & \mathrm{Re}(A) \end{bmatrix}.$$

Note that this matrix is symmetric, so by Theorem 5.2 we can use a generalized current flow embedding to compute $(THT^*)^+$.

By Property (P7) of Theorem 1.2.1 in [5], $(THT^*)^+ = TH^+T^*$. This implies that $H^+ = T^* \cdot (THT^*)^+ \cdot T$. The block structure of H implies that

$$H^+ = \begin{bmatrix} A^+ & 0 \\ 0 & \overline{A}^+ \end{bmatrix}.$$

Thus for any Hermitian matrix A, it is possible to transform the problem into a symmetric matrix problem, apply the generalized embedding technique, and transform back to get pseudoinverses of both A and its complex conjugate.

REFERENCES

[1] C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam, 1973.

[2] M. Bern, J. Gilbert, B. Hendrickson, N. Nguyen, and S. Toledo, *Support-graph preconditioners*, in Copper Mountain Conference on Iterative Methods, Copper Mountain, Colorado, 1998, pp. 1–7.

[3] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge, 1974. Second edition 1993.

[4] B. Bollobás, *Graph Theory: An Introductory Course*, Springer-Verlag, New York, 1979.

[5] S. L. Campbell and C. D. Meyer, Jr., *Generalized Inverses of Linear Transformations*, Dover, New York, 1991. Reprint of a work originally published by Pitman Publishing Limited (London) in 1979.

[6] A. K. Chandra, P. Raghavan, W. L. Ruzzo, R. Smolensky, and P. Tiwari, *The electrical resistance of a graph captures its commute and cover times*, in 21st Annual Symposium on Theory of Computation, Seattle, May 1989, ACM, pp. 574–586.

[7] P. Diaconis and D. Stroock, *Geometric bounds for eigenvalues of Markov chains*, Annals of Applied Probability, 1 (1991), pp. 36–61.

[8] M. Fiedler, *A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory*, Czechoslovak Mathematical Journal, 25 (1975), pp. 619–633.

[9] K. Gremban, G. Miller, and M. Zagha, *Performance evaluation of a parallel preconditioner*, in 9th International Parallel Processing Symposium, Santa Barbara, April 1995, IEEE, pp. 65–69.

[10] K. D. Gremban, *Combinatorial Preconditioners for Sparse, Symmetric, Diagonally Dominant Linear Systems*, PhD thesis, Carnegie Mellon University, Pittsburgh, October 1996. CMU CS Tech Report CMU-CS-96-123.

[11] S. Guattery, *Graph embedding techniques for bounding condition numbers of incomplete factor preconditioners*, Tech. Report 97-47, ICASE, September 1997. Submitted to Num. Lin. Alg. App.

[12] S. Guattery, F. T. Leighton, and G. Miller, *The Path Resistance Method for bounding λ_2 of a Laplacian*, in 8th ACM-SIAM Symposium on Discrete Algorithms, New Orleans, January 1997, ACM-SIAM.

[13] S. Guattery and G. L. Miller, *Embeddings and laplacian eigenvalues*, (1997). Submitted for publication as an ICASE Report. Submitted to SIAM J. Mat. Anal. App.

[14] M. Jerrum and A. Sinclair, *Approximating the permanent*, SIAM Journal on Computing, 18 (1989), pp. 1149–1178.

[15] N. Kahale, *A semidefinite bound for mixing rates of Markov chains*, in Proc. Fifth Integer Programming and Combinatorial Optimization Conference, 1996, pp. 190–203. Lecture Notes in Computer

Science 1084.

[16] A. Sinclair, *Improved bounds for mixing rates of Markov chains and multicommodity flow*, Combinatorics, Probability and Computing, 1 (1992), pp. 351–370.

[17] A. Sinclair and M. Jerrum, *Approximate counting, uniform generation and rapidly mixing Markov chains*, Information and Computation, 82 (1989), pp. 93–133.

[18] G. Strang, *Introduction to Applied Mathematics*, Wellesley Cambridge Press, Wellesley, MA, 1986.