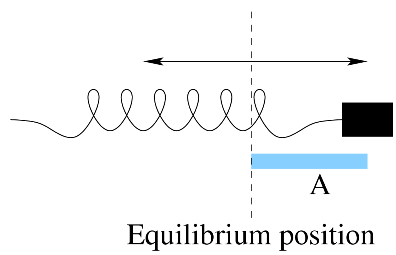
Physics 212 Classical and Modern Physics Spring 2008

Phasors & Phasor Diagrams

No, they're not little hand-held devices that can stun and kill unsuspecting aliens. At least, those aren't the ones we're talking about here. Phasors are a graphical representation of oscillating motion, and once you understand them a little, they provide for rather intuitive and straightforward way to handle wave interference problems.

1 A Phasor Diagram for a Single Oscillation

Consider a simple oscillating system, like a mass on a spring moving back and forth on a frictionless horizontal surface.



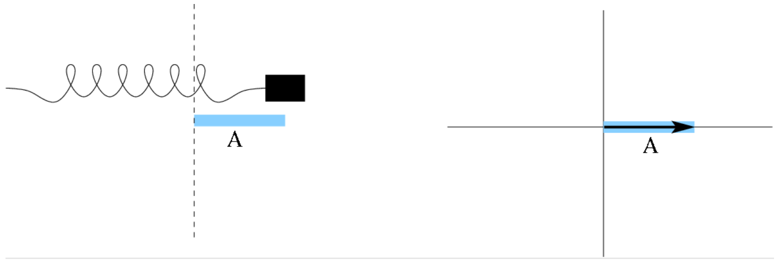
We can write an expression for the displacement of the mass as follows:

$$x(t) = A\cos\omega t$$

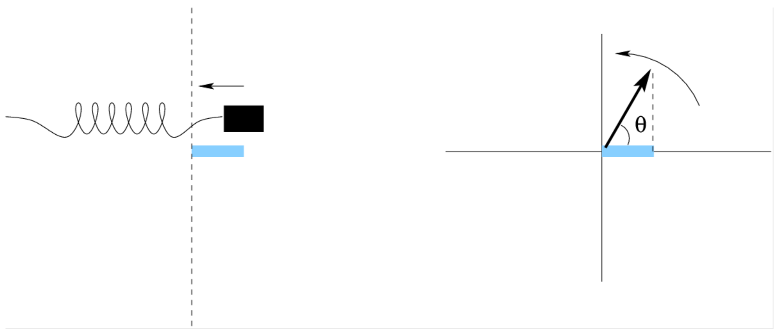
where A is the amplitude of the oscillation, and ω is the angular frequency in units of rad/s. That means that A is the value of the largest deviation (plus or minus) of the mass from its equilibrium position, and the value of ω governs the time it takes for the mass to complete one oscillation (recall that the period of the oscillation, $T = \frac{2\pi}{\omega}$, so a small ω means a long period while a large ω means a short period). Together, A and ω completely define the oscillation.

We can represent these two quantities, and therefore the oscillation itself, with a phasor diagram. A phasor is nothing more than a vector with magnitude is A that *rotates* with angular velocity ω. The concept of a rotating vector is a little strange, but it works out really well for oscillations.

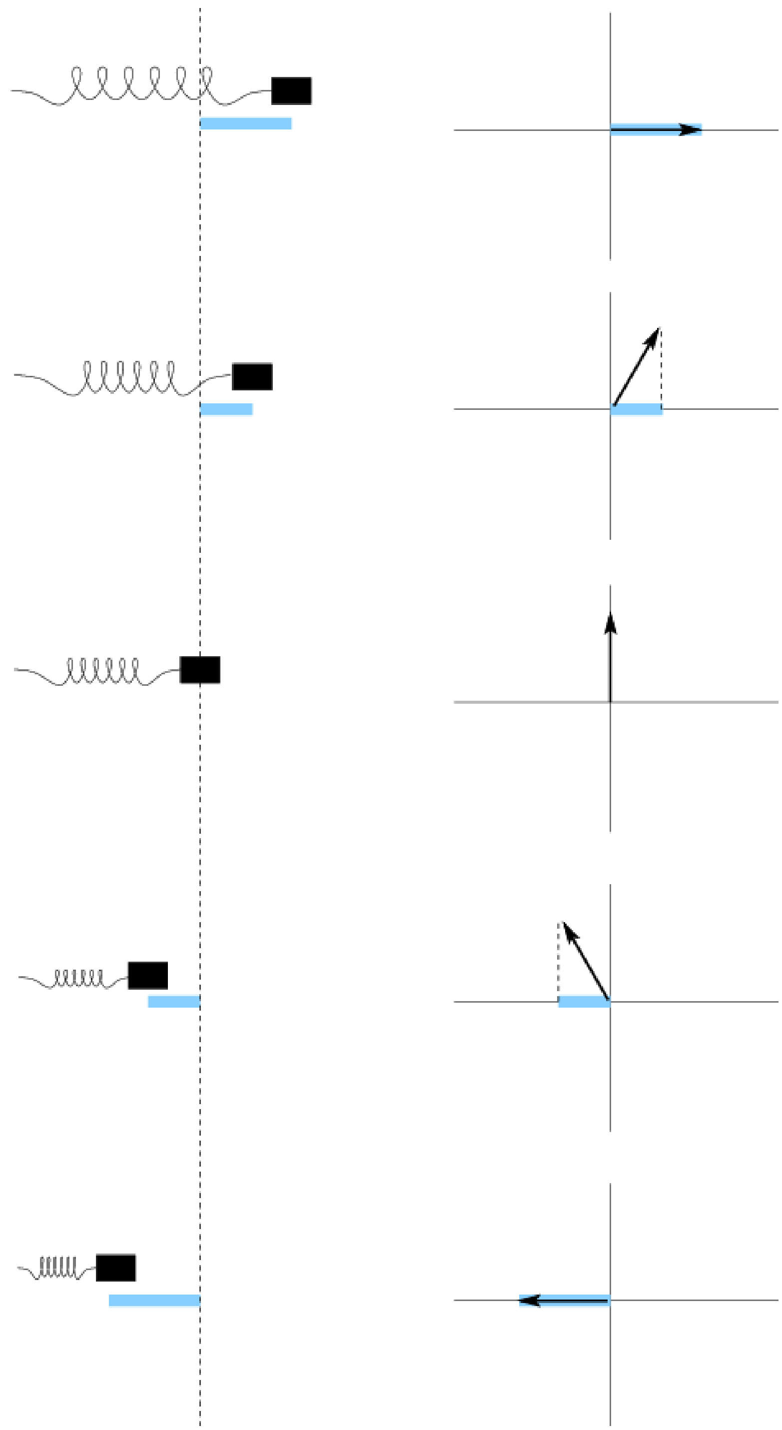
For the oscillation defined above $x(0) = A$, since $\cos 0 = 1$. We can represent the oscillation at this time with a phasor of length A that lies on the horizontal axis.



Not very interesting yet, I know, but here's the cool part: let time advance. As time passes, the mass moves back toward its equilibrium position. At the same time, the phasor rotates in the counterclockwise direction. The phasor's magnitude doesn't change, but the projection of the phasor onto the horizontal axis gets smaller.



Now if $\theta = \omega t$ in the phasor diagram, then the projection of the phasor on the horizontal axis is $A \cos \theta = A \cos \omega t$, and that's precisely the expression for the displacement of the mass at time t. This means that if the phasor rotates with angular velocity ω, its projection on the horizontal axis will always describe the displacement of the oscillation.



Note that the phasor does not change length; rather, it's just the projection of the phasor onto the horizontal axis that changes to describe the oscillation's changing displacement. This diagram should also give you a better idea of why we talk about "angular frequency" for oscillations. We're matching up the frequency of the oscillation with the angular velocity of the rotating phasor.

Examples

Ex. 1: Imagine you're at the beach and you walk into the surf until you're waist-deep in the water. You feel the waves passing by you, and you realize that you can express the change in the level of the water at your location with the expression

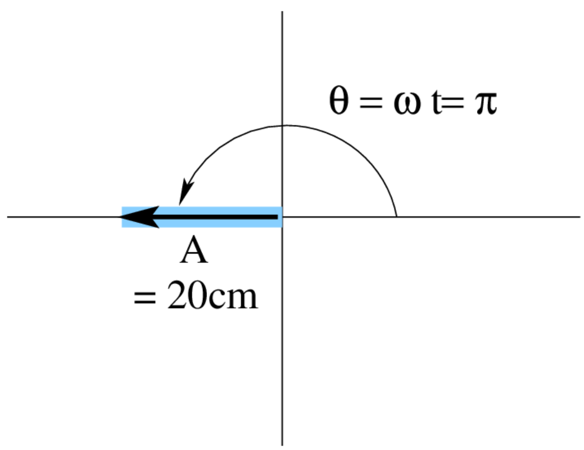
$$y(t) = A\cos\omega t$$

where $A = 20\,\text{cm}$ and $\omega = \frac{\pi}{3}\,\text{rad/s}$. Let's draw a phasor diagram to depict the level of the water at time $t = 3\,\text{s}$.

The phasor describing this oscillation will have magnitude $A = 20\,\text{cm}$, so we need a vector of that length. At time $t = 3\,\text{s}$, the phasor will have rotated by

$$\theta = \omega t = \left(\frac{\pi}{3}\right)(3) = \pi\,\text{rad}.$$

So therefore, our phasor diagram looks like this:



The projection of the phasor onto the horizontal axis is $-20\,\text{cm}$, so at this time, the wave height is at a minimum (i.e., a trough).

Important Note: Even though this problem is about an oscillation in the vertical direction, I still measure the displacement of the oscillation by the projection of the phasor onto the *horizontal* axis. There is no $x-$ or $y-$axis on a phasor diagram, so don't go and try to match up the orientation of the oscillation with the phasor diagram.

Ex. 2: A fire alarm goes off in your dorm. You run outside and stand in the cold weather waiting for the fire department to arrive and turn off the alarm (it's yet another false alarm). While you're waiting, you get out your pocket oscilloscope and measure the change in density of the air at your location due to the compression from the sound waves from the fire alarm. You measure the density change to be described by the following expression:

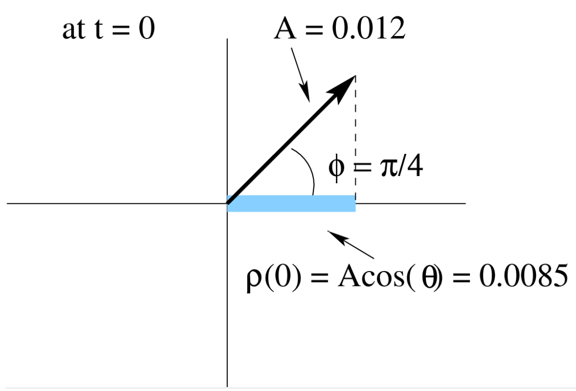
$$\rho(t) = A\cos\left(\omega t + \phi\right)$$

where $A = 0.012\,\mathrm{kg/m^3}$, $\omega = 5,000\pi\,\mathrm{rad/s}$, and $\phi = \frac{\pi}{4}$. You decide to draw a phasor diagram describing the density at time $t = 0.1\,\mathrm{ms}$.

This problem is a little trickier because of the non-zero ϕ inside the cosine. Conceptually, it means that when you decided to define time $t = 0$, the oscillation wasn't at a maximum. Instead, at time $t = 0$,

$$\rho(t) = A\cos\left(\phi\right) = (0.012)\cos\left(\frac{\pi}{4}\right) = 0.0085\,\mathrm{kg/m^3}.$$

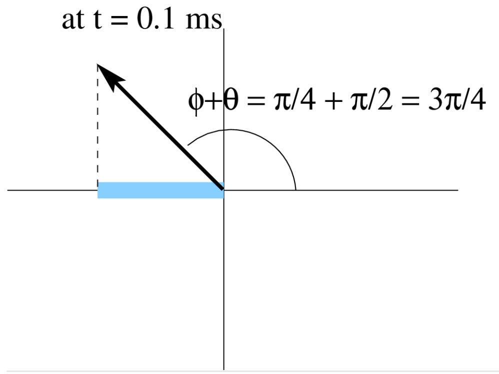
There's nothing wrong with this, but it will impact how we draw our phasor diagram. The amplitude of the oscillation is still $A = 0.012\,\mathrm{kg/m^3}$, so that must be the length of our phasor. However, if the projection of this phasor onto the horizontal axis at time $t = 0$ is to be $0.0085\,\mathrm{kg/m^3}$, then the phasor must be rotated. By how much? By ϕ.



The above picture is the "starting point" for our phasor diagram. To produce the phasor diagram for $t = 0.1\,\mathrm{ms}$, we need to let this phasor rotate for that time interval. It will rotate by

$$\theta = \omega t = (5000\pi)(0.0001) = \frac{\pi}{2}$$

from this starting point. So our phasor diagram will look like this:



The projection of the phasor at this time is

$$A\cos\left(\frac{3\pi}{4}\right) = -0.0085\,\mathrm{kg/m^3}$$

and so the density is lower than its equilibrium value due to the sound wave.

Problem A106 - Phasor Diagrams

Draw phasor diagrams depicting the oscillations described below at the noted times:

a) For the oscillation $x(t) = 3\cos\left(\frac{\pi}{3}t\right)$, draw phasor diagrams for $t = 0, 1, 2, 3, 5,$ and $6\,\mathrm{s}$.

b) For the oscillation $x(t) = 3\cos\left(\frac{\pi}{3}t + \frac{\pi}{3}\right)$, draw phasor diagrams for $t = 0, 1, 2, 3, 5,$ and $6\,\mathrm{s}$.

c) Consider the water waves described in **Ex. 1** above, but imagine that $t = 0$ corresponds to the moment when the water surface returns to its equilibrium position just after a crest passes. Draw phasor diagrams for the displacement of the water for times $t = 0, 1, 2, 3, 5,$ and $6\,\mathrm{s}$.