PHYSICS 331 Advanced Classical Mechanics

Final Examination

Thursday, December 17, 2009

Instructions:

- Attempt ALL questions.
- Show all work CLEARLY.
- Begin each question on a new page.
- Where necessary, take $g = 10$ m/s^2.

Problem 1

In Einstein's theory of relativity, the relation for the energy of a moving object is given by

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m is the mass of the object, v is the speed of the object relative to the observer, and c is the speed of light in vacuum. Expand this expression to the lowest order in v to obtain an approximate expression for the energy at non-relativistic speeds.

Problem 2

A particle moves subject to the potential energy

$$U(x) = U_o \left(\frac{a}{x} + \frac{x}{a} \right)$$

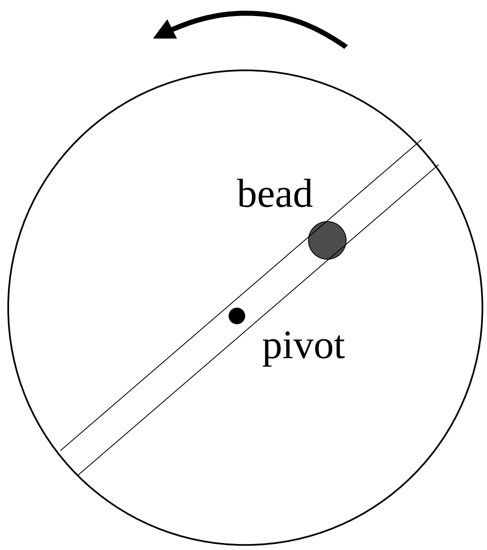
where U_o and a are positive.

(a) Locate all equilibrium points.

(b) Determine the stability for each equilibrium point.

(c) Obtain the frequency of small oscillations about the points where the equilibrium is stable.

Problem 3
A disk with a moment of inertia I can spin freely about a pivot, as shown in the figure below. There is a rail going all the way across the disk on which a small bead with mass m can slide freely. The bead cannot leave the rail. With the bead near the centre of the disk, the disk is started spinning; after that, there are no external forces or torques acting on the disk or the bead, although (of course) there are forces between the bead and the track on which it is moving. Consider the disk to be oriented horizontally, so that gravity is not relevant.



(a) Write down the Lagrangian of the system comprising the disk and the bead.

(b) Extract the (two) equations of motion using Lagrange's equations. Note that you do not need to solve the differential equations, just simply write them down.

(c) Each of the two differential equations in part (b) can be interpreted in terms of physical principles we have encountered in this course. Using a sentence or two, describe the physical principle to which each of the equations corresponds, and show (in detail) how the terms in the equation match up with that principle.

(d) Describe in words what will happen to the bead as the disk. Show how this arises from the equations of motion.

Problem 4
Describe what is meant by a 'central force,' and show that an object subject to a central force will maintain constant angular momentum.

Problem 5

A uniform rod of length b stands vertically upright on a rough floor and then tips over. Determine the rod's angular velocity just as it hits the floor. Be sure to include all calculations, including moments of inertia, in full detail.

Problem 6

Determine the ratio of the radius R to the height H of a right-circular cylinder of fixed volume V that minimises the surface area, A.

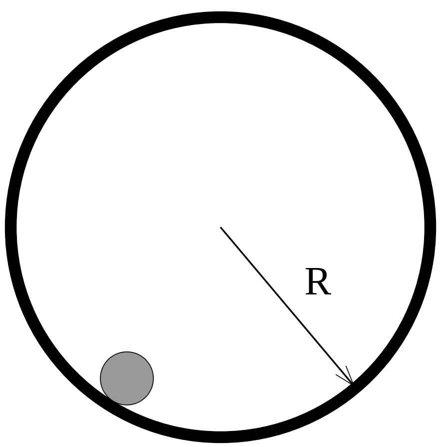
Problem 7

A hoop of mass m and radius R rolls without slipping down an inclined plane of mass M, which makes an angle α with the horizontal. Extract the Lagrangian for the system. Be sure to clearly identify your choice of generalised coordinates.

Problem 8

EITHER

Consider a hoop with a radius R and mass m that is free to roll without slipping on the floor. A bead, also of mass m, is confined to move around the hoop, but can slide freely with no friction.



OR

A double pendulum comprising a mass m suspended by a massless rod of length L from a fixed pivot, and a second mass M suspended by a massless rod of length L from the first mass m.

For your given choice:

(a) Write down the Lagrangian for the system.

(b) Determine the characteristic frequencies of the system for the case where $M = m$ and small oscillations.

(c) Determine the normal modes associated with the characteristic frequencies.

Problem 9
Consider the map

$$x_{n+1} = A\sqrt{x_n(1 - x_n)}$$

where $0 < x < 1$ and A ranges from 0 to 2.

Use a computer to simulate this map (Excel, Mathematica, etc.) and answer the following questions:

(a) Determine the approximate values of A for which the behaviour bifurcates from period-1 to period-2 behaviour. Repeat for period-2 to period-4 behaviour.

(b) Make plots of iterative values of x, i.e. x versus n for a few representative values of A between 0 and 2. Be sure to include at least one plot showing

  (i) period-1 behaviour;

  (ii) period-2 behaviour;

  (iii) period-4 behaviour;

  (iv) chaotic behaviour.

Problem 10
The motor of a speed boat is shut off just as it attains a speed of v_o. Now, the boat is slowed down by a resistive force $F_r = Ce^{-kv}$, where C and k are positive constants. Determine how long it will take the boat to stop and how far it will have travelled before it stops. Measure the distance from the point where the motor stopped.

Problem 11
Halley's comet has a period of revolution around the Sun of 76 years.

(a) Determine the semi-major axis of its orbit. Give your answer in AU, i.e., an astronomical unit defined as the mean distance between the Earth and the Sun over one Earth orbit.

(b) The comet's minimum distance of approach to the Sun is 0.60 AU, measured from the comet to the centre of the Sun. Determine the maximum distance between the comet and the Sun in the course of its revolution.

(c) Determine the eccentricity of the trajectory of Halley's comet.

Problem 12

Answer the following as either True or False:

- The trajectories of phase space can never cross.
- ALL "chaotic" systems (as defined by physicists) show sensitivity to initial conditions.
- A system is sensitive to initial conditions if there is a positive Lyapunov exponent.
- When using Lagrangian (undetermined) multipliers, the smallest number of degrees of freedom is appropriate for the problem must be used.
- Variational approaches are useful if you are looking for a function that minimises some quantity.
- Theoretically, it is only possible to find a set of principle axes if rotation if the object has at least one axis of symmetry.
- If an object is spinning with constant angular velocity, then there is no external torque acting on that object.
- If a gyroscope is prevented from precessing, it will fall over.
- For a system with N degrees of freedom, an appropriate phase space will also have N dimensions.

Problem 13

Describe a concept or technique that we covered in this course that you found particularly interesting and/or valuable. Give reasons for your choice; examples may be helpful.