Problem

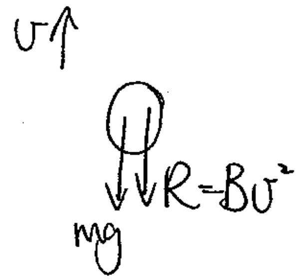
A particle is projected vertically upwards with v_0.

let $R = Bv^2$ be nature of fluid friction.

To determine the speed of the particle when it returns to its initial position, need to determine distance travelled vertically before stopping. This is necessary because $\Sigma \vec{F}$ is different for the paths up and down.

up

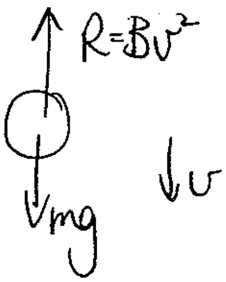
$+\uparrow$



$\Sigma F = ma$

$\therefore -mg - Bv^2 = ma$ (1)

down



$\downarrow +$

$\Sigma F = ma$

$\therefore mg - Bv^2 = ma$ (2)

note: when $R = mg$, velocity is terminal

ie $Bv_T^2 = mg$ (3)

To determine the total distance travelled upwards:

(1) $\rightarrow$ $-mg - Bv^2 = ma$; using $a = v\dfrac{dv}{dx}$ (since want to know x when $v=0$)

$$\therefore mv\frac{dv}{dx} = -mg - Bv^2$$

$$\therefore mv\frac{dv}{dx} = -Bv_T^2 - Bv^2 \quad \text{from (3)}$$

$$\therefore \frac{v\,dv}{(v^2+v_T^2)} = -\frac{B}{m}dx$$

integrating both sides from initial $\begin{pmatrix} x=0 \\ v=v_0 \end{pmatrix}$ to final $\begin{pmatrix} x=x_{max} \\ v=0 \end{pmatrix}$

$$\int_i^f \frac{v\,dv}{(v^2+v_T^2)} = -\int_i^f \frac{B}{m}\,dx$$

$$\therefore \frac{1}{2}\int_i^f \frac{2v\,dv}{(v^2+v_T^2)} = -\frac{B}{m}\left[x\right]_i^f$$

$$\therefore \frac{1}{2}\left[\ln(v^2+v_T^2)\right]_i^f = -\frac{B}{m}(x-0)$$

$$\therefore \ln(v^2+v_T^2) - \ln(v_0^2+v_T^2) = \frac{-2Bx}{m}$$

Setting $v=0$, $x=x_{max}$

$$\therefore \ln\left(\frac{v_T^2}{v_0^2+v_T^2}\right) = \frac{-2Bx_{max}}{m}$$

$$\therefore x_{max} = \frac{m}{2B}\ln\left(\frac{v_0^2+v_T^2}{v_T^2}\right) \quad (4)$$

: total distance travelled upwards

On the way down, want to know the speed of the particle after travelling a distance x_{max} downwards.

Using eqn (2)

taking $x=0$ at top of flight $\begin{pmatrix} v_i=0 \\ x_i=0 \end{pmatrix}$

$$mv\frac{dv}{dx} = mg - Bv^2 \quad \text{, from (2)}$$

$$\therefore mv\frac{dv}{dx} = Bv_T^2 - Bv^2 \quad \text{, using (3)}$$

$$\therefore \frac{v\,dv}{(v^2-v_T^2)} = \frac{-B}{m}\,dx$$

integrating both sides from initial to final

$$\frac{1}{2}\int_i^f \frac{2v\,dv}{(v^2-v_T^2)} = \frac{-B}{m}\int_i^f dx$$

$$\therefore \left[\ln(v^2 - v_T^2)\right]_i^f = \frac{-2B}{m}(x_f - x_i)$$

setting $v_i = 0$ $v_f = v$
$x_i = 0$ $x_f = x_{max}$

$$\therefore \ln(v^2 - v_T^2) - \ln(-v_T^2) = \frac{-2B}{m}(x_{max} - 0)$$

$$\therefore \ln\left(\frac{v^2 - v_T^2}{-v_T^2}\right) = \frac{-2B}{m} \cdot x_{max} \quad ; \quad x_{max} = \frac{m}{2B}\ln\left(\frac{v_0^2 + v_T^2}{v_T^2}\right)$$

from (4)

$$\therefore \ln\left(\frac{v^2 - v_T^2}{-v_T^2}\right) = \frac{-\cancel{2B}}{\cancel{m}} \cdot \frac{\cancel{m}}{\cancel{2B}} \cdot \ln\left(\frac{v_0^2 + v_T^2}{v_T^2}\right)$$

$$\therefore \ln\left(\frac{v^2 - v_T^2}{-v_T^2}\right) = \ln\left(\frac{v_T^2}{v_0^2 + v_T^2}\right)$$

Equating the arguments gives,

$$\frac{v^2 - v_T^2}{-v_T^2} = \frac{v_T^2}{v_0^2 + v_T^2}$$

$$\therefore v^2 = v_T^2 - \frac{v_T^4}{v_0^2 + v_T^2}$$

$$\therefore v^2 = \frac{v_T^2(v_0^2 + v_T^2) - v_T^4}{v_0^2 + v_T^2} = \frac{v_0^2 v_T^2 + \cancel{v_T^4} - \cancel{v_T^4}}{v_0^2 + v_T^2}$$

$$\therefore v = \frac{v_0 v_T}{\sqrt{v_0^2 + v_T^2}}$$

, the speed of the particle when it returns to its initial position.

, as required.