PHYSICS 331 Advanced Classical Mechanics
Problem Set 23

Task 1

Consider a two particle system under the influence of a conservative central force. Let the masses of the two particles be m_1 and m_2, located at distance $\vec{r_1}$ and $\vec{r_2}$ relative to some origin, O.

(a) Show that of the force is conservative, the potential energy U can be written as $U(\vec{r_1} - \vec{r_2})$.

(b) Show that because the force is central, then $U(\vec{r_1} - \vec{r_2}) = U(r)$ where $\vec{r} = \vec{r_1} - \vec{r_2}$.

(c) Choose the generalised coordinates of relative position ($\vec{r}$) and centre-of-mass position ($\vec{R}$):

$$\vec{r} = \vec{r_1} - \vec{r_2} \quad \text{and} \quad \vec{R} = \frac{\sum_i m_i \vec{r_i}}{\sum_i m_i}$$

show that

$$\vec{r_1} = \vec{R} + \left(\frac{m_2}{M}\right)\vec{r} \quad \text{and} \quad \vec{r_2} = \vec{R} - \left(\frac{m_1}{M}\right)\vec{r}$$

where $M = m_1 + m_2$, the total mass.

(d) Now, the Lagrangian of the motion can be written as

$$\mathcal{L} = \frac{1}{2}m_1\dot{\vec{r_1}}^2 + \frac{1}{2}m_2\dot{\vec{r_2}}^2 + U(r).$$

Show that, using the generalised coordinates, the Lagrangian transforms to

$$\mathcal{L} = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 + U(r).$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass.

The Lagrangian was now be expressed as

$$\mathcal{L} = \mathcal{L}_{cm} + \mathcal{L}_{rel}.$$

(e) Explain the value of transforming to the centre-of-mass frame.

Task 2

Central forces and angular momentum.

(a) Show that the angular momentum ($\mathbf{L}$) is conserved for a two-body system under the influence of a central force.

(b) Show that

$$\vec{L} = \vec{r_1} \times \vec{p_1} + \vec{r_2} \times \vec{p_2}$$

reduces to

$$\vec{L} = \vec{r} \times \mu\dot{\vec{r}}.$$

i.e., the total angular momentum can be thought of as similar to a single particle of reduced mass (μ) and position $\vec{r}$.

(c) Explain why the position vector $\vec{r}$ lies in a plane and this may be treated as a 2-dimensional vector.

Task 3
Using the Lagrangian in part (d) of Task 1, extract the equations of motion from both the R and r equations.

(a) Show that the R equation yields conservation of momentum.

(b) Show that the r equation yields $\mu \ddot{\vec{r}} = -\vec{\nabla} U(r)$.

Now, express the relative coordinate $\vec{r}$ in terms of polar coordinates: r and ϕ.

(c) Show that the Lagrangian, $\mathcal{L}_{rel}$ can be written as

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r).$$

(d) Extract the equations of motion for both r and ϕ showing that

$$\mu r\dot{\phi}^2 - \frac{\partial U}{\partial r} = \mu\ddot{r} \qquad \text{and} \qquad \dot{\phi} = \frac{\ell}{\mu r^2}$$

where ℓ is the angular momentum of the system as measured in the centre-of-mass frame.

Task 4

(a) Show that if we write $U_{cfg} = \frac{\ell^2}{2\mu r^2}$, then the r equation in Task 3 can be expressed as

$$\mu\ddot{r} = -\frac{dU_{eff}}{dr}, \qquad \text{where} \qquad U_{eff} = U(r) + U_{cfg}.$$

(b) Show that if we write $E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}$, then

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}(E - U(r)) - \frac{\ell^2}{\mu^2 r^2}}$$

(c) Beginning with the first equation of motion in part (d) of Task 3, make the change of variable $u = \frac{1}{r}$ and show that the equation can be expressed as

$$\frac{d^2u}{d\phi^2} = -u - \frac{\mu}{\ell^2 u^2}F,$$

where $F = -\frac{\partial U}{\partial r}$.

Other Things
You will find that re-reading sections 8.1–8.5 may be helpful in performing these exercises.