PHYSICS 331 ADVANCED CLASSICAL MECHANICS
Problem Set 27

The Foucault Pendulum.
The Foucault pendulum is made of a very heavy mass m suspended by a light wire from a tall ceiling. The pendulum is allowed to swing freely for long periods of time, and moves in both the east–west and north–south directions.

Task 1
Show that, in the rotating frame of the Earth, the equation of motion of the mass is:

$$m\ddot{\mathbf{r}} = \mathbf{T} + m\mathbf{g}_{\mathbf{o}} + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} + 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega}.$$

As described in class, the second and third terms on the right side can be combined to give $m\mathbf{g}$, where $\mathbf{g}$ is the observed free-fall acceleration, making the equation of motion:

$$m\ddot{\mathbf{r}} = \mathbf{T} + m\mathbf{g} + 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega}.$$

Task 2
Let's now choose a set of axes such that x is east, y is north, and z is vertically upwards (in the direction of $\mathbf{g}$). The figure below shows such a case. Show that in the case of small oscillations, where the angle β between the pendulum and the vertical is always small,

$$T_z = T\cos\beta \approx T \quad \text{and} \quad T \approx mg,$$

and also that

$$T_x = -\frac{mgx}{L} \quad \text{and} \quad T_y = -\frac{mgy}{L}.$$

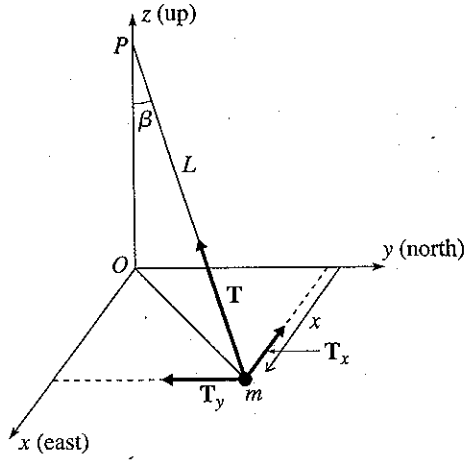


Figure 9.16 A Foucault pendulum comprises a bob of mass m suspended by a light wire of length L from the point P on a high ceiling. The tension force on the bob is shown as $\mathbf{T}$ and its x and y components are T_x and T_y. For small oscillations the angle β is very small.

Task 3

Writing the vectors:

$$\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z}) \qquad \text{and} \qquad \mathbf{\Omega} = (0, \Omega \sin\theta, \Omega\cos\theta)$$

show that

$$\dot{\mathbf{r}} \times \mathbf{\Omega} = (\dot{y}\ \Omega\cos\theta - \dot{z}\ \Omega\sin\theta, -\dot{x}\ \Omega\cos\theta, \dot{x}\ \Omega\sin\theta).$$

Task 4

(a) Explain why, in the case of the pendulum, $\dot{z}$ is appreciably smaller that either $\dot{x}$ or $\dot{y}$.

(b) Using the results in Task 3 to substitute for the second expression in Task 1, show that in this case where $\dot{z}$ is small compared with $\dot{x}$ and $\dot{y}$, that

$$\ddot{x} = \frac{gx}{L} + 2\dot{y}\Omega\cos\theta$$

$$\ddot{y} = \frac{gy}{L} - 2\dot{x}\Omega\cos\theta$$

Task 5

Setting $\frac{g}{L} = \omega_o^2$ and replacing $\Omega\cos\theta$ with Ω_z, the z-component of the Earth's angular velocity, the above equations reduce to the set of coupled equations:

$$\ddot{x} - 2\Omega_z\dot{y} + \omega_o^2 x = 0$$

$$\ddot{y} + 2\Omega_z\dot{x} + \omega_o^2 y = 0.$$

We can use an elegant approach from complex analysis to solve these coupled equations. We define

$$\eta = x + iy.$$

If we think of the $x - y$ plane as the complex plane, then this complex number contains the same information as the position in the $x - y$ plane, and a plot of η is actually a bird's eye view of the pendulum projected onto the $x - y$ plane.

Show that the two coupled equations now reduce to:

$$\ddot{\eta} + 2i\Omega_z\dot{\eta} + \omega_o^2\eta = 0.$$

Task 6

We will try/guess the solution of this differential equation to be of the form $\eta(t) = e^{i\alpha t}$. Show that this can only be a solution if

$$\alpha^2 - 2\Omega_z\alpha - \omega_o^2 = 0,$$

or

$$\alpha = \Omega_z \pm \sqrt{\Omega_z^2 + \omega_o^2}.$$

which can be approximated to

$$\alpha = \Omega_z \pm \omega_o$$

since the Earth's angular velocity Ω is so much smaller than the pendulum's ω_o.

Task 7

Thus, the general solution to the differential equation describing the pendulum can be written as

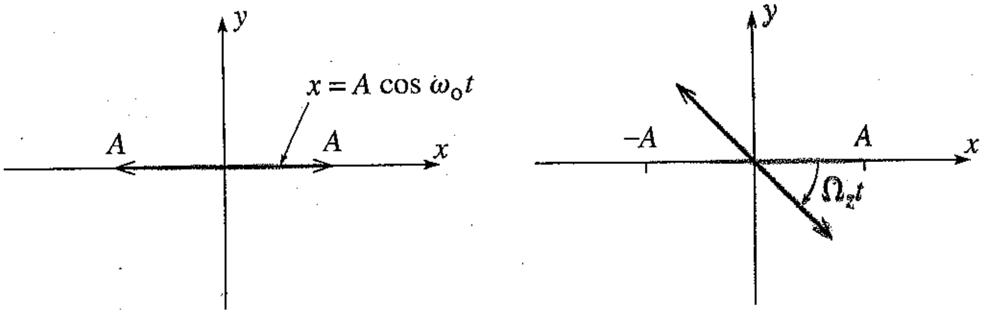
$$\eta(t) = e^{-i\Omega_z t}\left(C_1 e^{i\omega_o t} + C_2 e^{-i\omega_o t}\right).$$

The parameters C_1 and C_2 can be determined by the initial conditions. Suppose at $t = 0$, the pendulum was pulled aside in the x-direction (east) to a position $x = A$ and $y = 0$, and is released from rest ($v_{xo} = v_{yo} = 0$). Show that $C_1 = C_2 = A/2$, and that the solutions becomes

$$\eta(t) \equiv x(t) + iy(t) = Ae^{-i\Omega_z t}\cos\omega_o t.$$

Task 8

Consider the nature of the solution; note that the amplitude of the pendulum's oscillations is a complex exponential, dependent on Ω_z. And since we have cast the complex plane to be consistent with the $x - y$ plane, this tells us that the plane in which the pendulum oscillates about its equilibrium position is governed by this amplitude. See the figure below.



(a) $t \approx 0$ (b) $t > 0$

(a) Confirm that, initially, the pendulum swing with simple harmonic motion along the x-axis with a frequency of ω_o between $-A$ and A.

(b) Compare the frequency at which the pendulum swings to the frequency at which the plane of oscillation changes under conditions on Earth.

(c) Show that the plane of oscillation rotates in a clockwise direction in the northern hemisphere and counter-clockwise in the southern hemisphere.

(d) At which locations on the Earth does the plane of oscillations *not* rotate?

(e) For a Foucault pendulum constructed at Bucknell, determine how long it would take a long, well-constructed pendulum to change its plane of oscillation by 90°?