PHYSICS 331 ADVANCED CLASSICAL MECHANICS
Problem Set 32

Problem 1
Taylor: Chapter 10, Problem 39. HINT: Read through example 10.3 of Taylor.

Problem 2
Taylor: Chapter 10, Problem 42.

Problem 3
Taylor: Chapter 10, Problem 45.

Problem 4
Taylor: Chapter 10, Problem 51.

Other Things
We will continue our journey with Coupled Oscillations. Read Marion and Thornton's chapter 12, sections 12.1 through 12.3. Please remember to submit your journal entry.
Consider a top consisting of a uniform cone spinning freely about its tip at 1800 rpm.

\[ H = \text{height of top} = 10\text{cm} \]
\[ R = \text{base radius} = 2.5\text{cm} \]

To determine the angular frequency (\( \Omega \)) at which it precesses, we note that the precessional frequency of a top due to a weak torque is given by

\[ \Omega_{\text{precess}} = \frac{Mg r}{\frac{3}{10} MR^2} \]

where \( L_e = \frac{1}{2} \omega \)

and \( \frac{3}{10} MR^2 \) for a cone

(see Taylor 10.39)

\[ W = 1800 \text{ rpm} \]
\[ = 1800 \cdot \frac{2\pi}{60} \text{ rad/sec} \]

\[ Mg r = \text{net torque} \]
\[ \Rightarrow r = \text{distance from top's pivot to its centre-of-mass} \]
\[ = \frac{3}{4} H \text{ for a cone.} \]

\[ \therefore \Omega_{\text{precess}} = \frac{Mg \left(\frac{3}{4} H\right)}{\frac{3}{10} MR^2 \cdot W} = \frac{\frac{3}{4} \cdot 10 \cdot 9 \cdot H}{R^2 \cdot W} \]

\[ \therefore \Omega_{\text{precess}} = \frac{5}{2} \cdot (9.8) (0.10) \]
\[ \frac{(0.025)^2 \cdot 1800 \cdot 2\pi/60}{(0.025)^2 \cdot 1800 \cdot 2\pi/60} \]
\[ = 20.80 \text{ rad/sec} \]
\[ \approx 199 \text{ rpm.} \]
Consider a book that is 30cm x 20cm x 3cm, held closed by a rubber band. It is then thrown up into the air spinning about an axis that is close to the book's shortest symmetry axis at 180 rpm.

\[ a = 30\text{cm} \]
\[ b = 20\text{cm} \]
\[ c = 3\text{cm} \]

To determine the angular frequency of the small oscillations about its axis of rotation, recall, the inertia tensor is given by

\[ \mathbf{I} = \frac{m}{12} \begin{pmatrix}
  b^2+c^2 & 0 & 0 \\
  0 & a^2+c^2 & 0 \\
  0 & 0 & a^2+b^2
\end{pmatrix} \]

for rotation about the centre of mass.

To determine the angular freq. about the short axis \( \Rightarrow \) want \( \Omega \) about \( \omega_3 \).

From (10.91), we showed

\[ \Omega^2 = \frac{(\lambda_3 - \lambda_2)(\lambda_5 - \lambda_1)}{\lambda_1 \lambda_2} \omega_3^2 \]

\[ \lambda_1 = b^2+c^2 \]
\[ \lambda_2 = a^2+c^2 \]
\[ \lambda_3 = a^2+b^2 \]

\[ 2 \Rightarrow \Omega^2 = \frac{\left[ (a^2+b^2)-(a^2+c^2) \right] \left[ (a+b)^2-(b^2+c^2) \right]}{(b^2+c^2)(a^2+c^2)} \omega_3^2 \]

\[ = \frac{(b^2-c^2)(a^2-c^2)}{(b^2+c^2)(a^2+c^2)} \omega_3^2 \]

\[ \Omega^2 = \frac{(20^2-3^2)(30^2-3^2)}{(20^2+3^2)(30^2+3^2)} \left( 180 \text{rpm} \right)^2 \]
\[ S_2 = \sqrt{0.037} W_3 = 0.968 W_3 = 0.968 \times (180 \text{ rpm}) = 174 \text{ rpm}. \]

If spinning about an axis close to the longest axis, we follow the same procedure as above except we interchange 1->3 for rotation about a drum.

\[ \therefore \Omega^2 = \frac{(\lambda_1 - \lambda_2) (\lambda_1 - \lambda_3)}{\lambda_3 \lambda_2} W_3^2 \]

\[ = \frac{(c^2 - a^2) (b^2 - a^2)}{c^4 + a^4} W_3^2 \]

\[ \therefore \Omega = \sqrt{0.037} W_3 = 0.614 W_3 = 110.5 \leq 111 \text{ rpm}. \]
Because of the Earth's equatorial bulge, its moment about the polar axis is slightly greater than the other two moments:
\[ \lambda_1 = \lambda_2, \text{ but } \lambda_3 \approx 1.00327 \lambda_1. \]

(a) The free precession of this body (the Earth) is given by

\[ \Omega_b = \frac{\lambda_1 - \lambda_3}{\lambda_1} \omega_3 \]

\[ \hat{e}_3 \text{ is the polar axis} \Rightarrow \omega_3 = 1 \text{ rev/day} = 2\pi \text{ rad/day} \]

\[ \therefore \Omega_b = \frac{\lambda_1 - 1.00327 \lambda_1}{\lambda_1} \omega_3 \]

\[ \Omega_b = (0.00327) \cdot \omega_3 = \frac{2\pi}{T_b} \]

\[ \therefore T_b = \frac{2\pi}{\Omega_b} = \frac{2\pi}{0.00327 \cdot \omega_3} = \frac{2\pi}{0.00327 \cdot 2\pi/\text{day}} \]

\[ = \frac{1}{0.00327} \text{ days} \]

\[ \therefore T_b = 305.8 \text{ days} \approx 306 \text{ days, as required.} \]

(b) The angle between \( \omega \) and the polar axis = 0.2 arc sec = 0.2 \( (\frac{1}{60} \times \frac{1}{60}) \) degrees.

\[ \Omega_8 = \frac{\omega \sqrt{\lambda_3^2 + (\lambda_2^2 - \lambda_3^2)}}{\lambda_1} \approx \frac{\omega}{\lambda_1} \sqrt{\lambda_3^2 + 0} \]

\[ = \frac{\lambda_3 \omega}{\lambda_1} \]

\[ \approx 1 \omega \gg 1 \text{ rev/day}. \]