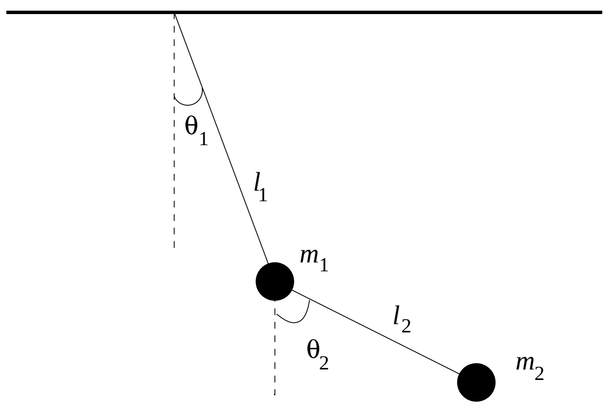
PHYSICS 331 Advanced Classical Mechanics
Problem Set 34

The Double Pendulum

Task 1 **Using the Lagrangian to obtain the equations of motion.**

Consider a double pendulum, comprising a mass m_1 suspended by a massless rod of length ℓ_1 from a fixed pivot point, and a second mass m_2 suspended by a massless rod of ℓ_2 from m_1, as shown in the figure below.



Taking the zero of potential energy to be at the equilibrium position, show that the potential energy of the system to be

$$U = (m_1 + m_2)\ g\ \ell_1(1 - \cos\theta_1) + m_2\ g\ \ell_2 \cos\theta_2.$$

Task 2

Show that the total kinetic energy of the system is given by

$$T = \frac{1}{2}(m_1 + m_2)\ \ell_1^2\ \dot{\theta_1}^2 +\ m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\ +\ \frac{1}{2}\ \ell_2^2\ \dot{\theta_2}^2.$$

Task 3

Making the small angle approximation, show that the kinetic energy of the system reduces to

$$T = \frac{1}{2}(m_1 + m_2)\ \ell_1^2\ \dot{\theta_1}^2 +\ m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2 +\ \frac{1}{2}\ \ell_2^2\ \dot{\theta_2}^2$$

and the potential energy of the system reduces to

$$U = \frac{1}{2}\ (m_1 + m_2)\ g\ \ell_1\ \theta_1^2 + \frac{1}{2}\ m_2\ g\ \ell_2\ \theta_2^2.$$

Task 4

Using the Lagrangian, show that the Lagrange equation for the generalised coordinate θ_1 is

$$(m_1 + m_2)\ \ell_1^2\ \ddot{\theta_1}^2\ +\ m_2\ \ell_1\ \ell_2\ \ddot{\theta}_2\ =\ -\ (m_1 + m_2)\ g\ \ell_1\ \theta_1$$

and the Lagrange equation for the generalised coordinate θ_2 is

$$m_2\ \ell_1\ \ell_2\ \ddot{\theta}_1\ +\ m_2\ \ell_2^2\ \ddot{\theta}_2\ =\ -\ m_2\ g\ \ell_2\ \theta_2.$$

Task 5

Re-write the two equations for θ_1 and θ_2 as a single matrix equation of the form

$$\mathbf{M}\ddot{\theta}\ =\ -\mathbf{K}\theta$$

where we introduce the column of generalised coordinates

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

Show that the two matrices are

$$\mathbf{M} = \begin{pmatrix} (m_1 + m_2)\ \ell_1^2 & m_2\ \ell_1\ \ell_2 \\ m_2\ \ell_1\ \ell_2 & m_2\ \ell_2^2 \end{pmatrix} \quad \text{and} \quad \mathbf{M} = \begin{pmatrix} (m_1 + m_2)\ g\ \ell_1 & 0 \\ 0 & m_2\ g\ \ell_2 \end{pmatrix}.$$

Task 6 **Examining the symmetric case.**
Solving for the normal modes of the double pendulum in general is algebraically messy and not particularly illuminating. As such, we will examine the symmetric case where $m_1 = m_2 = m$ and $\ell_1 = \ell_2 = \ell$.

To search for normal modes of oscillation, we will look for solutions of the form $\ddot{\theta} = -\omega^2 \theta$. In matrix form, show that this translates to: $\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0$ and the two possible solutions yields

$$\omega_1^2 = (2 - \sqrt{2})\ \omega_o^2 \quad \text{and} \quad \omega_2^2 = (2 + \sqrt{2})\ \omega_o^2$$

where $\omega_o = \sqrt{g/\ell}$ is the frequency of a single pendulum of length ℓ, again for small angles.

Task 7
Now, the general solution for $\theta_1(t)$ and $\theta_2(t)$ can be written as a linear combination of the two normal modes, $\eta_1(t)$ and $\eta_2(t)$.

In general, we can write

$$\theta_1(t) = a_{11} \cos(\omega_1 t - \delta_1)\ +\ a_{12} \cos(\omega_2 t - \delta_2) =\ a_{11}\ \eta_1(t)\ + a_{12}\ \eta_2(t)$$
$$\theta_2(t) = a_{21} \cos(\omega_1 t - \delta_1)\ +\ a_{22} \cos(\omega_2 t - \delta_2) =\ a_{21}\ \eta_1(t)\ + a_{22}\ \eta_2(t).$$

or

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$

To obtain the normal modes, we are interested in the case where each coordinate oscillates at only *one* of the two modes, with constant amplitude.

To find the first normal mode (ω_1), we are looking for the case where

$$\theta_1(t) = a_{11} \cos(\omega_1 t - \delta_1) \quad \text{and} \quad \theta_2(t) = a_{21} \cos(\omega_1 t - \delta_1)$$

or

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \cos(\omega_1 t - \delta_1)$$

Substitute the normal mode frequency ω_1 into the matrix equation $(\mathbf{K} - \omega^2 \mathbf{M})\theta = 0$ to show that $a_{21} = \sqrt{2} a_{11}$. Also, show that for the second normal mode, ω_2, $a_{22} = -\sqrt{2} a_{12}$.

Task 8
Combining the information about the two normal modes, show that the first mode corresponds to the two pendula moving exactly in phase with the amplitude of the lower pendulum being $\sqrt{2}$ times that of the upper pendulum. Obtain the conditions for the normal mode oscillations for the second mode, showing that the pendula oscillate exactly out-of-phase, again with the lower pendulum's amplitude being $\sqrt{2}$ of the upper one.

Problem 2
Thornton and Marion: Chapter 12, Problem 17.