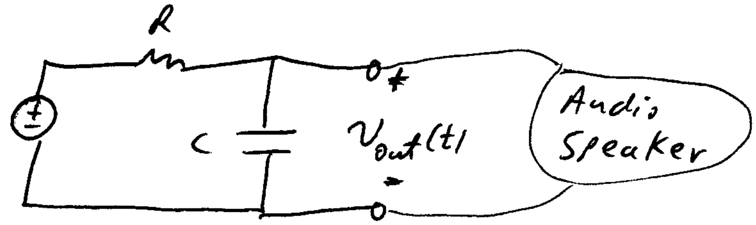
Sinusoidal Response of Circuits
(Frequency selective "filters")

Relevant reading in the text:

Class demonstration to motivate "filters":

$v_{in}(t) = 1.\cos\omega t$



$R = 1.1\,k\Omega$

$C = .22\,\mu F$

Vary ω, the frequency of input sine wave.

Low frequencies: (Below $f = 658$ Hz)

v_{out} is about the same size as v_{in}, and tone is audible.

High frequencies: (Above $f = 658$ Hz)

v_{out} is smaller than v_{in}, and tone is less loud.

Our goal is to understand what is happening in this circuit.

Such "filters" are used in many applications: Tune radios, TVs, treble + bass on stereo, equalizers, Dolby for cassettes, compact disks, ...

Basic idea: Circuit is a voltage divider.

The capacitor "impedance" decreases as ω increases, so v_{out} gets smaller as ω increases.

Outline:

- we have reviewed sine waves on a separate set of notes.
- we will define "phasors", which describe sine waves by a complex number.
- we will review the arithmetic of complex numbers
- we will define "impedance", which is like a resistance that varies with frequency,
- we will apply phasors + impedance to circuits.

Phasors:

A phasor characterizes the amplitude + phase of a sinusoidal time signal:

Time signal: $v(t) = A \cos[\omega t + \varphi]$

Phasor : $\underline{v} = A \angle \varphi = A e^{j\varphi}$

The frequency ω is <u>omitted</u> from the phasor.

The rationale is that "linear" circuits (the kind we've been studying) do not change the frequency ω — they only change the amplitude + phase of a sine wave.

[Recall the RC demo on p. 1; the output has the same frequency as the input, but it can be smaller in amplitude + shifted in phase.]

Examples:

Time Signal	Phasor
$v(t) = 10.1 \cos(100t)$	$\underline{v} =$
$v(t) = 2 \cos(5t + 30^\circ)$	$\underline{v} =$
$v(t) = \quad \cos(77t \quad)$	$\underline{v} = 4 \angle -35^\circ$

We manipulate phasors as complex numbers.

Recall "imaginary" numbers: $\boxed{j = \sqrt{-1}}$

(You may have used $i = \sqrt{-1}$ in math — we use j since i is used for current)

Imaginary numbers arise in solving equations like $x^2 = -4 \Rightarrow x = j2$

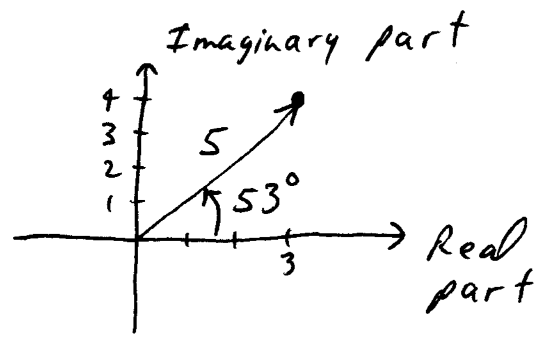
Because: $(j2)^2 = j^2 \cdot 2^2 = (\sqrt{-1})^2 \cdot 4 = -4$

So $\boxed{j^2 = -1}$ $\Rightarrow$ remember that.

"Complex" numbers have a real part and an imaginary part.

Ex: $3 + j4$

We can visualize this as a vector in the "complex plane":

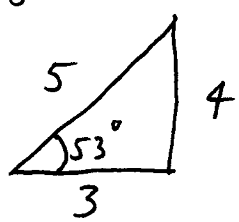


$3 + j4$ is the "rectangular form".

An equivalent representation is the "polar form":

$5 \angle 53^\circ$ = vector of length 5 at angle 53° with real axis.

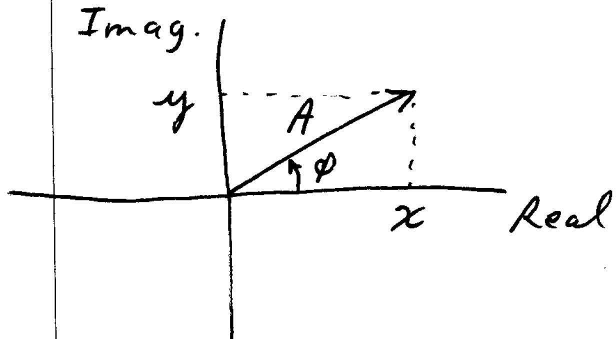
How I got this:



$$5 = \sqrt{3^2 + 4^2} = \sqrt{9+16}$$

$$53^\circ = \tan^{-1} \frac{4}{3}$$

General conversion between rectangular + polar forms:



$$x + jy$$
$$= A \angle \varphi$$

If know $x + jy$:

$$\boxed{A = \sqrt{x^2 + y^2} = \text{"magnitude"}}$$

$$\varphi = \tan^{-1} \frac{y}{x} \quad \left\{ \text{Need to be careful about quadrant} \right.$$

If know $A \angle \varphi$:

$$x = A \cos \varphi$$
$$y = A \sin \varphi$$

A phasor $A \angle \varphi$ is like a vector, with one additional feature: phasors can be <u>multiplied</u> in a way vectors cannot.

Complex number arithmetic:

Addition + subtraction: Operate on real + imaginary parts separately.

$$(3+j4) + (-1+j2) = (3-1) + j(4+2)$$
$$= 2 + j6$$

Use rectangular form to add/subtract.

Multiplication: with rectangular form, use algebra, + remember $j^2 = -1$:

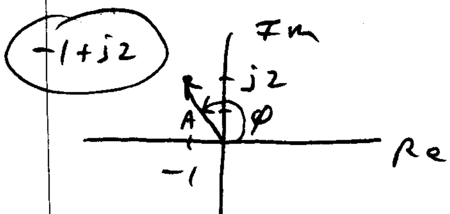
$$(3+j4)\cdot(-1+j2) = (3)(-1) + (3)(j2) + (j4)(-1) + (j4)(j2)$$
$$= -3 + j6 + j(-4) + j^2(8)$$
$$= -3 - 8 + j(6-4) = -11 + j2$$

Multiplication + Division are simpler with polar form:

$$(A_1 \angle \phi_1)\cdot(A_2 \angle \phi_2) = A_1 \cdot A_2 \angle(\phi_1 + \phi_2)$$

$$\frac{A_1 \angle \phi_1}{A_2 \angle \phi_2} = \frac{A_1}{A_2} \angle(\phi_1 - \phi_2)$$

For example above: $3 + j4 = 5 \angle 53°$



$A = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\phi = 180° - \tan^{-1}\frac{2}{1} = 117°$

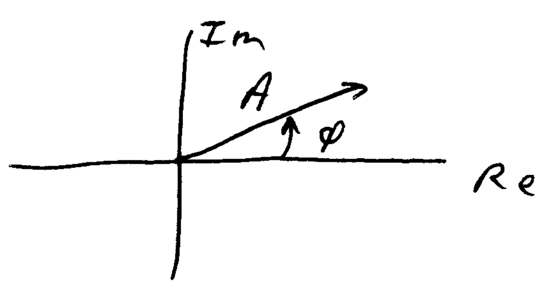
$$(3+j4)\cdot(-1+j2) = (5\angle 53^\circ)\cdot(\sqrt{5}\angle 117^\circ)$$
$$= 5\cdot\sqrt{5}\angle 170^\circ$$
$$= 5\cdot\sqrt{5}\cos 170^\circ + j\,5\sqrt{5}\sin 170^\circ$$
$$= -11 + j2,$$ which agrees.

Your calculator may do complex number arithmetic — feel free to use those capabilities.

So we will think of phasors as complex numbers, and manipulate them as complex numbers.

Time function: $v(t) = A\cos(\omega t + \varphi)$

phasor: $\underline{v} = A\angle\varphi$



This simplifies the analysis — it is easier than solving differential eqs!

Impedance

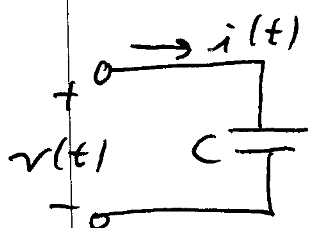
Recall Ohm's law for resistors:

$$R = \frac{v(t)}{i(t)}$$

The current + voltage are <u>always</u> related this way for a resistor, even if the voltage + current varies with time. (For example, $v(t) = 2\cos(100t)$.)

We want to get a similar relation for a capacitor. We will take the ratio of voltage + current <u>phasors</u>.

Consider $v(t) = A\cos\omega t$ across cap.:

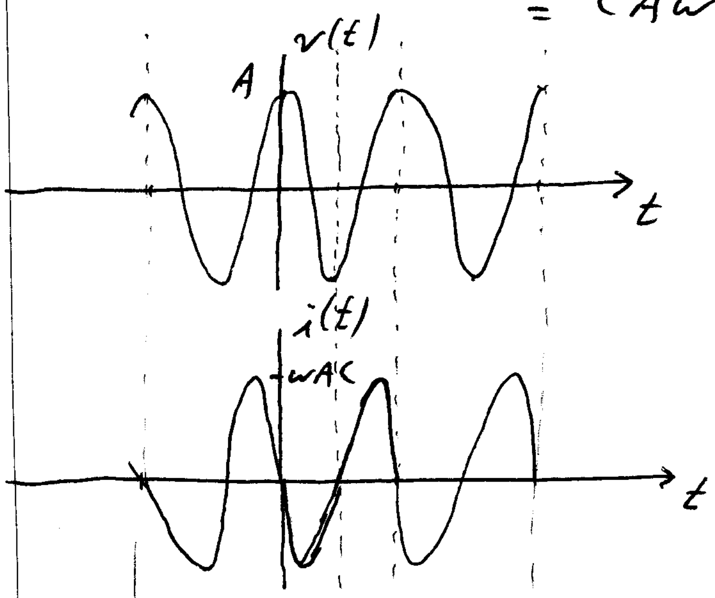


$$i(t) = C\frac{dv}{dt}$$

$$= C\cdot\frac{d}{dt}A\cos\omega t$$

$$= -CA\omega\sin\omega t$$

$$= CA\omega\cos[\omega t + 90°]$$

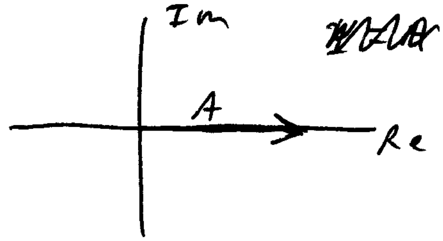


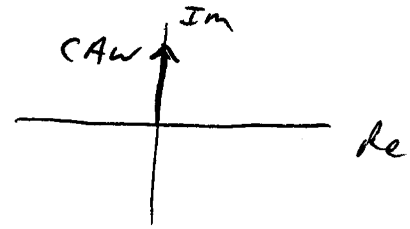
Current is scaled in amplitude by factor ωC, and shifted in phase by 90°, relative to voltage.

Phasors for voltage + current:

$v(t) = A \cos \omega t$ $\qquad\qquad$ $i(t) = CA\omega \cos[\omega t + 90^\circ]$

$\underline{v} = A \angle 0^\circ$ $\qquad\qquad$ $\underline{i} = CA\omega \angle 90^\circ$





$\underline{v} = A$ $\qquad\qquad$ $\underline{i} = j(CA\omega)$

Note the phasors $\underline{v}$ and $\underline{i}$ convey the amplitude + phase difference between voltage + current.

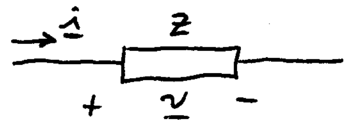
Define impedance of capacitor:

$$Z_c = \frac{\underline{v}}{\underline{i}} = \frac{A}{jCA\omega} = \boxed{\frac{1}{j\omega C}}$$

- This is like a resistance: it relates voltage + current flow, like Ohm's law.
- Note the impedance varies with freq. ω. In fact, it gets smaller as ω increases, which agrees with the demo. on page 1.
- The impedance is a complex number, which allows it to account for the phase shift between $v(t)$ and $i(t)$.

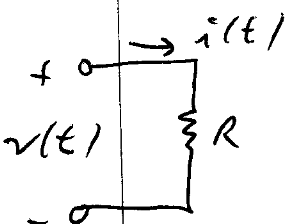
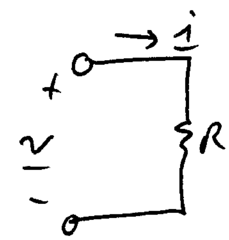
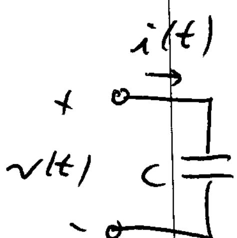
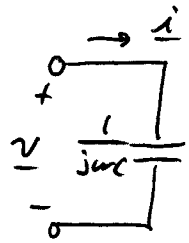
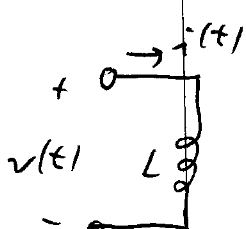
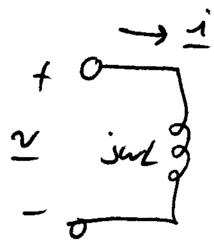
Ohm's Law for phasors + impedance:

$$\boxed{\underline{v} = \underline{i}\ Z}$$

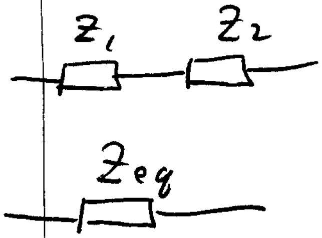


Here $\underline{v}$, $\underline{i}$, and Z are complex numbers.

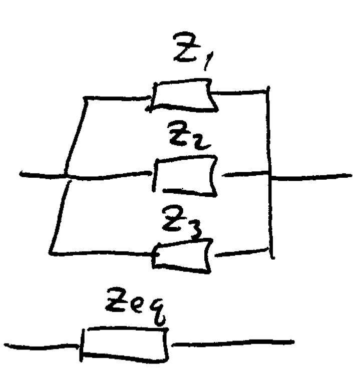
Impedance of resistors, capacitor, inductor:

General Formulas		Phasors + impedance for Sinusoidal signals	
	$v(t) = R\,i(t)$		$\underline{v} = R\,\underline{i}$
	$i(t) = C\dfrac{dv}{dt}$ $v(t) = \dfrac{1}{C}\displaystyle\int_{-\infty}^{t} i(x)\,dx$		$\underline{v} = \left(\dfrac{1}{j\omega C}\right)\underline{i}$
	$v(t) = L\dfrac{di}{dt}$ $i(t) = \dfrac{1}{L}\displaystyle\int_{-\infty}^{t} v(x)\,dx$		$\underline{v} = (j\omega L)\,\underline{i}$

- Can transform circuit elements to impedances
- Impedances in series/parallel combine like resistors

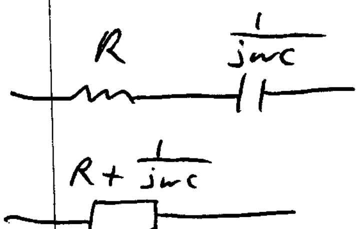


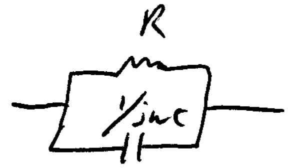
$$Z_{eq} = Z_1 + Z_2$$



$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

Examples:





$$\frac{R \cdot \frac{1}{j\omega C}}{R + 1/j\omega C} = \frac{R}{j\omega RC + 1}$$

The actual impedance will depend on the values of R, C, and the frequency ω of the applied sine wave.

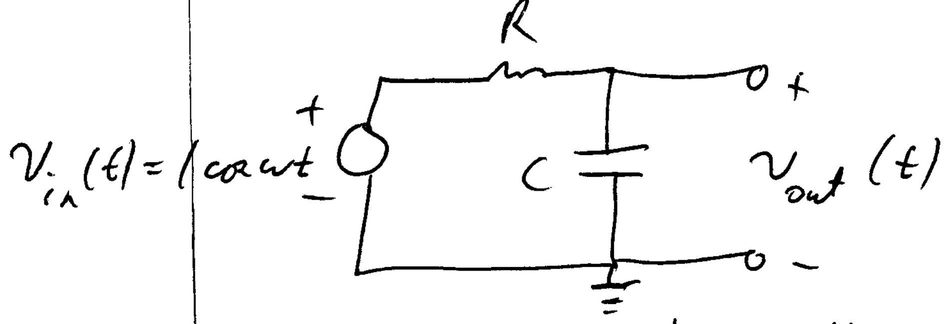
$$\boxed{\text{The impedance } \underline{\text{changes}} \text{ with Freq. } \omega.}$$

All of our circuit analysis tools apply to circuits with impedances & phasors:

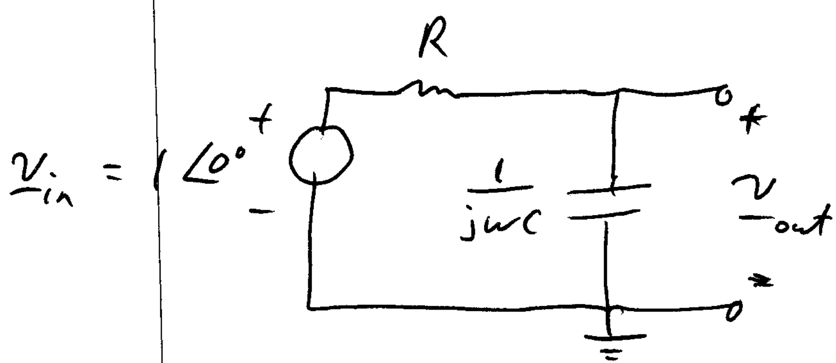
- Generalized Ohm's Law: $\underline{V} = \underline{I}\,\underline{Z}$
- KVL, KCL
- Voltage divider
- Nodal & mesh analysis
-

Thevenin
Superposition
Op amps
...

Now we can understand the RC circuit on page 1:



Redraw circuit with phasors & impedances – we will then be able to solve for the output voltage phasor, for a given input freq. ω.



Treat impedances like resistors:

$\underline{v}_{out} =$