ELEC 320
Prof. Rich Kozick

Notes for Homework 10

Problem: Prove that for a continuous-time system that is *linear* and *time-invariant*, the zero-state response (ZSR) of the system to a sinusoidal input is a sine wave with the same frequency as the input wave, but a different amplitude and phase shift. Also, find an expression for the *frequency response* $H(\omega)$ of the system in terms of the *impulse response* $h(t)$.

Use the following approach.

1. Please explain why it is true that the ZSR of any linear, time-invariant (LTI) system is completely described by the impulse response $h(t)$ of the system. (Are there any LTI systems for which this is not true?) If the impulse response $h(t)$ is known, then the system output $y(t)$ due to any input $x(t)$ is given by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\, d\lambda$$

2. Now consider a particular input $x(t) = \cos(\omega_0 t)$ that is applied to a LTI system with impulse response $h(t)$. Put this $x(t)$ into the convolution integral, and look at the resulting $y(t)$. You should be able to recognize that $y(t)$ is a sine wave with the same frequency ω_0, but with a different amplitude and phase shift. The trigonometric identities at the bottom of the page will be helpful.

3. In terms of the frequency response of the system $H(\omega)$, recall that we expect that the system output has the form

$$y(t) = |H(\omega_0)| \cos(\omega_0 t + \angle H(\omega_0)).$$

   Use your result from item 2 to relate the frequency response $H(\omega)$ of the system to the impulse response $h(t)$. This provides a mathematical connection between the frequency domain and time domain descriptions of a system.

4. You now understand the very important result that a sine wave input to a LTI system produces a sine wave output with the same frequency but different amplitude and phase shift!

Here are some useful identities:

$$\cos[\omega_0(t-\lambda)] = \cos(\omega_0 t)\cos(\omega_0\lambda) + \sin(\omega_0 t)\sin(\omega_0\lambda)$$

$$A\cos(\omega_0 t) - B\sin(\omega_0 t) = H\cos(\omega_0 t + \theta)$$

where $H = \sqrt{A^2 + B^2}$ and $\theta = \arctan(B/A)$.