Signal Processing and Linear Systems

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Preface

This book presents a comprehensive treatment of signals and linear systems suitable for juniors and seniors in electrical engineering. The book contains most of the material from my earlier popular book *Linear Systems and Signals* (1992) with added chapters on analog and digital filters and digital signal processing. There are also additional applications to communications and controls. The sequence of topics in this book is somewhat different from the earlier book. Here, the Laplace transform follows Fourier, whereas in the 1992 book, the sequence was the exact opposite. Moreover, the continuous-time and the discrete-time are treated sequentially, whereas in the 1992 book, both approaches were interwoven. The book contains enough material in discrete-time systems so that it can be used not only for a traditional course in *Signals and Systems*, but also for an introductory course in *Digital Signal processing*.

A perceptive author has said: "The function of a teacher is not so much to cover the topics of study as to uncover them for the students." The same can be said of a textbook. This book, as all my previous books, emphasizes the physical appreciation of concepts rather than mere mathematical manipulation of symbols. There is a temptation to treat an engineering subject, such as this, as a branch of applied mathematics. This view ignores the physical meaning behind various results and derivations, which deprives a student of intuitive understanding of the subject. I have used mathematics not so much to prove an axiomatic theory as to enhance the physical and intuitive understanding. Wherever possible, theoretical results are interpreted heuristically† and are supported by carefully chosen examples and analogies.‡

Notable Features

The notable features of the book include the following:

1. Emphasis on intuitive and heuristic understanding of the concepts and physical meaning of mathematical results leading to deeper appreciation and easier comprehension of the concepts. As one reviewer put it, "One thing I found very appealing about this book is great balance of mathematical and intuitive explanation." Most reviewers of the book have noted the reader friendly character of the book with unusual clarity of presentation.
2. The book provides extensive applications in the areas of communication, controls, and filtering.
3. For those who like to get students involved with computers, computer solutions of several examples are provided using MATLAB®, which is becoming a

†Heuristic [Greek *heuriskein*, to invent, discover]: a method of education in which the pupil is trained to find out things for himself. The word 'Eureka' (I have found it) is the 1st pers. perf. indic. act., of heuriskein.

‡If these lines appear familiar to you, there is a good reason. I have used them in the preface of some of my earlier books, including *Signals, Systems, and Communication* (Wiley, 1965). What is more interesting, many other authors also have borrowed them for their preface.

standard software package in an electrical engineering curriculum.

4. Many students are handicapped by an inadequate background in basic material such as complex numbers, sinusoids, sketching signals, Cramer's rule, partial fraction expansion, and matrix algebra. I have added a chapter that addresses these basic and pervasive topics in electrical engineering. Response by student has been unanimously enthusiastic.
5. There are over 200 worked examples along with exercises (with answers) for students to test their understanding. There are also about 400 selected problems of varying difficulty at the end of the chapters. Many problems are provided with hints to steer a student in the proper direction.
6. The discrete-time and continuous-time systems are covered sequentially, with flexibility to teach them concurrently if so desired.
7. The summary at the end of each chapter proves helpful to students in summing up essential developments in the chapter, and is an effective tool in the study for tests. Answers to selected problems are helpful in providing feedback to students trying to assess their knowledge.
8. There are several historical notes to enhance student's interest in the subject. These facts introduce students to historical background that influenced the development of electrical engineering.

Organization

The book opens with a chapter titled *Background*, which deals with the mathematical background material that a student taking this course is expected to have already mastered. It includes topics such as complex numbers, sinusoids, sketching signals, Cramer's rule, partial fraction expansion, matrix algebra. The next 7 chapters deal with continuous-time signals and systems followed by 5 chapters treating discrete-time signals and systems. The last chapter deals with state-space analysis. There are MATLAB examples dispersed throught the book. The book can be readily tailored for a variety of courses of 30 to 90 lecture hours. It can also be used as a text for a first undergraduate course in *Digital signal Processing* (DSP).

The organization of the book permits a great deal of flexibility in teaching the continuous-time and discrete-time concepts. The natural sequence of chapters is meant for a sequential approach in which all the continuous-time analysis is covered first, followed by discrete-time analysis. It is also possible to integrate (interweave) continuous-time and discrete-time analysis by using a appropriate sequence of chapters.

Credits

The photographs of Gauss (p. 3), Laplace (p. 380), Heaviside (p. 380), Fourier (p. 188), Michelson (p. 206) have been reprinted courtesy of the Smithsonian Institution. The photographs of Cardano (p. 3) and Gibbs (p. 206) have been reprinted courtesy of the Library of Congress. Most of the MATLAB examples were prepared by Dr. O. P. Mandhana of IBM, Austin, TX.

Acknowledgments

Several individuals have helped me in the preparation of this book. I am grateful for the helpful suggestions of the reviewers Professors. Dwight Day (Kansas State

University), Prof. Mark Herro (University of Notre Dame), Hua Lee (University of California, Santa Barbara), Tina Tracy (University of Missouri, Columbia), J.K. Tugnait (Auburn University), R.L. Tummala (Michigan State University) I owe Dr. O.P. Mandhana a debt of gratitude for his helpful suggestions and his painstaking solutions to most of the MATLAB problems. Special thanks go to Prof. James Simes for generous help with computer solution of several problems. I am much obliged to Ing Ming Chang for his enthusiastic and crucial help in solving MATLAB problems and using computer to prepare the manuscript. Finally I would like to mention the enormous but invisible sacrifices of my wife Rajani in this endeavor.

B. P. Lathi

MATLAB

Throughout this book, examples have been provided to familiarize the reader with computer tools for systems design and analysis using the powerful and versatile software package MATLAB. Much of the time and cost associated with the analysis and design of systems can be reduced by using computer software packages for simulation. Many corporations will no longer support the development systems without prior computer simulation and numerical results which suggest a design will work. The examples and problems in this book will assist the reader in learning the value of computer packages for systems design and simulation.

MATLAB is the software package used throughout this book. MATLAB is a powerful package developed to perform matrix manipulations for system designers. MATLAB is easily expandable and uses its own high level language. These factors make developing sophisticated systems easier. In addition, MATLAB has been carefully written to yield numerically stable results to produce reliable simulations.

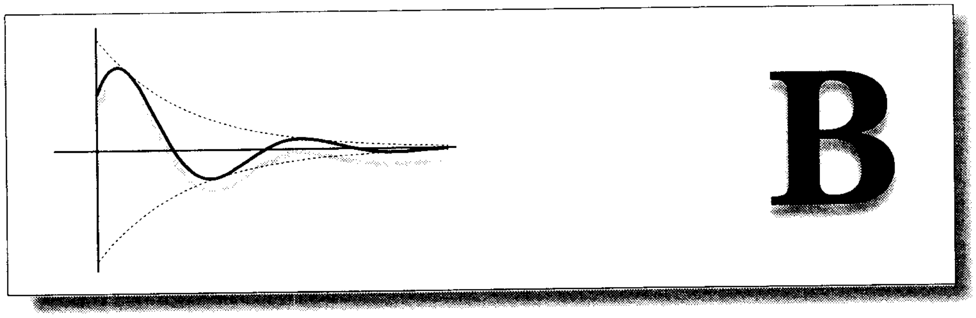
All the computer examples in this book are verified to be compatible with the student edition of the MATLAB when used according to the instructions given in its manual. The reader should make sure that \MATLAB\BIN is added in the DOS search path. MATLAB can be invoked by executing the command MATLAB. The MATLAB banner will appear after a moment with the prompt '>>'. MATLAB has a useful on-line help. To get help on a specific command, type HELP COMMAND NAME and then press the ENTER key. DIARY FILE is a command to record all the important keyboard inputs to a file and the resulting output of your MATLAB session to be written on the named file. MATLAB can be used interactively, or by writing functions (subroutines) often called M files because of the .M extension used for these files. Once familiar with the basics of MATLAB, the reader can easily learn how to write functions and to use MATLAB's existing functions.

The MATLAB M-files have been created to supplement this text. This includes all the examples solved by MATLAB in the text. These M-files may be retrieved from the Mathworks anonymous FTP site at

ftp://ftp.mathworks.com/pub/books/lathi/.

O. P. Mandhana

Contents



Background

The topics discussed in this chapter are not entirely new to students taking this course. You have already studied many of these topics in earlier courses or are expected to know them from your previous training. Even so, this background material deserves a review because it is so pervasive in the area of signals and systems. Investing a little time in such a review will pay big dividends later. Furthermore, this material is useful not only for this course but also for several courses that follow. It will also be helpful as reference material in your future professional career.

B.1 Complex Numbers

Complex numbers are an extension of ordinary numbers and are an integral part of the modern number system. Complex numbers, particularly **imaginary numbers**, sometimes seem mysterious and unreal. This feeling of unreality derives from their unfamiliarity and novelty rather than their supposed nonexistence! Mathematicians blundered in calling these numbers "imaginary," for the term immediately prejudices perception. Had these numbers been called by some other name, they would have become demystified long ago, just as irrational numbers or negative numbers were. Many futile attempts have been made to ascribe some physical meaning to imaginary numbers. However, this effort is needless. In mathematics we assign symbols and operations any meaning we wish as long as internal consistency is maintained. A healthier approach would have been to define a symbol i (with any term but "imaginary"), which has a property $i^2 = -1$. The history of mathematics is full of entities which were unfamiliar and held in abhorrence until familiarity made them acceptable. This fact will become clear from the following historical note.

B.1-1 A Historical Note

Among early people the number system consisted only of natural numbers (positive integers) needed to count the number of children, cattle, and quivers of arrows. These people had no need for fractions. Whoever heard of two and one-half children or three and one-fourth cows!

However, with the advent of agriculture, people needed to measure continuously varying quantities, such as the length of a field, the weight of a quantity of butter, and so on. The number system, therefore, was extended to include fractions. The ancient Egyptians and Babylonians knew how to handle fractions, but **Pythagoras** discovered that some numbers (like the diagonal of a unit square) could not be expressed as a whole number or a fraction. Pythagoras, a number mystic, who regarded numbers as the essence and principle of all things in the universe, was so appalled at his discovery that he swore his followers to secrecy and imposed a death penalty for divulging this secret.[1] These numbers, however, were included in the number system by the time of Descartes, and they are now known as **irrational numbers**.

Until recently, **negative numbers** were not a part of the number system. The concept of negative numbers must have appeared absurd to early man. However, the medieval Hindus had a clear understanding of the significance of positive and negative numbers.[2,3] They were also the first to recognize the existence of absolute negative quantities.[4] The works of **Bhaskar** (1114-1185) on arithmetic (*Līlāvatī*) and algebra (*Bījaganit*) not only use the decimal system but also give rules for dealing with negative quantities. Bhaskar recognized that positive numbers have two square roots.[5] Much later, in Europe, the banking system that arose in Florence and Venice during the late Renaissance (fifteenth century) is credited with developing a crude form of negative numbers. The seemingly absurd subtraction of 7 from 5 seemed reasonable when bankers began to allow their clients to draw seven gold ducats while their deposit stood at five. All that was necessary for this purpose was to write the difference, 2, on the debit side of a ledger.[6]

Thus the number system was once again broadened (generalized) to include negative numbers. The acceptance of negative numbers made it possible to solve equations such as $x+5=0$, which had no solution before. Yet for equations such as $x^2+1=0$, leading to $x^2=-1$, the solution could not be found in the real number system. It was therefore necessary to define a completely new kind of number with its square equal to -1. During the time of Descartes and Newton, imaginary (or complex) numbers came to be accepted as part of the number system, but they were still regarded as algebraic fiction. The Swiss mathematician **Leonhard Euler** introduced the notation i (for **imaginary**) around 1777 to represent $\sqrt{-1}$. Electrical engineers use the notation j instead of i to avoid confusion with the notation i often used for electrical current. Thus

$$j^2 = -1$$

and

$$\sqrt{-1} = \pm j$$

This notation allows us to determine the square root of any negative number. For example,

$$\sqrt{-4} = \sqrt{4} \times \sqrt{-1} = \pm 2j$$

When imaginary numbers are included in the number system, the resulting numbers are called **complex numbers**.

Origins of Complex Numbers

Ironically (and contrary to popular belief), it was not the solution of a quadratic equation, such as $x^2+1=0$, but a cubic equation with real roots that made

Gerolamo Cardano (left) and Karl Friedrich Gauss (right).

imaginary numbers plausible and acceptable to early mathematicians. They could dismiss $\sqrt{-1}$ as pure nonsense when it appeared as a solution to $x^2 + 1 = 0$ because this equation has no real solution. But in 1545, **Gerolamo Cardano** of Milan published *Ars Magna* (*The Great Art*), the most important algebraic work of the Renaissance. In this book he gave a method of solving a general cubic equation in which a root of a negative number appeared in an intermediate step. According to his method, the solution to a third-order equation†

$$x^3 + ax + b = 0$$

is given by

$$x = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} + \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

For example, to find a solution of $x^3 + 6x - 20 = 0$, we substitute $a = 6$, $b = -20$ in the above equation to obtain

$$x = \sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}} = \sqrt[3]{20.392} - \sqrt[3]{0.392} = 2$$

We can readily verify that 2 is indeed a solution of $x^3 + 6x - 20 = 0$. But when Cardano tried to solve the equation $x^3 - 15x - 4 = 0$ by this formula, his solution

†This equation is known as the *depressed cubic* equation. A general cubic equation

$$y^3 + py^2 + qy + r = 0$$

can always be reduced to a depressed cubic form by substituting $y = x - \frac{p}{3}$. Therefore any general cubic equation can be solved if we know the solution to the depressed cubic. The depressed cubic was independently solved, first by **Scipione del Ferro** (1465-1526) and then by **Niccolo Fontana** (1499-1557). The latter is better known in the history of mathematics as **Tartaglia** ("Stammerer"). Cardano learned the secret of the depressed cubic solution from Tartaglia. He then showed that by using the substitution $y = x - \frac{p}{3}$, a general cubic is reduced to a depressed cubic.

was

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

What was Cardano to make of this equation in the year 1545? In those days negative numbers were themselves suspect, and a square root of a negative number was doubly preposterous! Today we know that

$$(2 \pm j)^3 = 2 \pm j11 = 2 \pm \sqrt{-121}$$

Therefore, Cardano's formula gives

$$x = (2 + j) + (2 - j) = 4$$

We can readily verify that $x = 4$ is indeed a solution of $x^3 - 15x - 4 = 0$. Cardano tried to explain halfheartedly the presence of $\sqrt{-121}$ but ultimately dismissed the whole enterprise as being "as subtle as it is useless." A generation later, however, **Raphael Bombelli** (1526-1573), after examining Cardano's results, proposed acceptance of imaginary numbers as a necessary vehicle that would transport the mathematician from the *real* cubic equation to its *real* solution. In other words, while we begin and end with real numbers, we seem compelled to move into an unfamiliar world of imaginaries to complete our journey. To mathematicians of the day, this proposal seemed incredibly strange.[7] Yet they could not dismiss the idea of imaginary numbers so easily because this concept yielded the real solution of an equation. It took two more centuries for the full importance of complex numbers to become evident in the works of Euler, Gauss, and Cauchy. Still, Bombelli deserves credit for recognizing that such numbers have a role to play in algebra.[7]

In 1799, the German mathematician **Karl Friedrich Gauss**, at a ripe age of 22, proved the fundamental theorem of algebra, namely that every algebraic equation in one unknown has a root in the form of a complex number. He showed that every equation of the nth order has exactly n solutions (roots), no more and no less. Gauss was also one of the first to give a coherent account of complex numbers and to interpret them as points in a complex plane. It is he who introduced the term *complex numbers* and paved the way for general and systematic use of complex numbers. The number system was once again broadened or generalized to include imaginary numbers. Ordinary (or real) numbers became a special case of generalized (or complex) numbers.

The utility of complex numbers can be understood readily by an analogy with two neighboring countries X and Y, as illustrated in Fig. B.1. If we want to travel from City a to City b (both in Country X), the shortest route is through Country Y, although the journey begins and ends in Country X. We may, if we desire, perform this journey by an alternate route that lies exclusively in X, but this alternate route is longer. In mathematics we have a similar situation with real numbers (Country X) and complex numbers (Country Y). All real-world problems must start with real numbers, and all the final results must also be in real numbers. But the derivation of results is considerably simplified by using complex numbers as an intermediary. It is also possible to solve all real-world problems by an alternate method, using real numbers exclusively, but such procedure would increase the work needlessly.

B.1-2 Algebra of Complex Numbers

A complex number (a, b) or $a + jb$ can be represented graphically by a point whose Cartesian coordinates are (a, b) in a complex plane (Fig. B.2). Let us denote this complex number by z so that

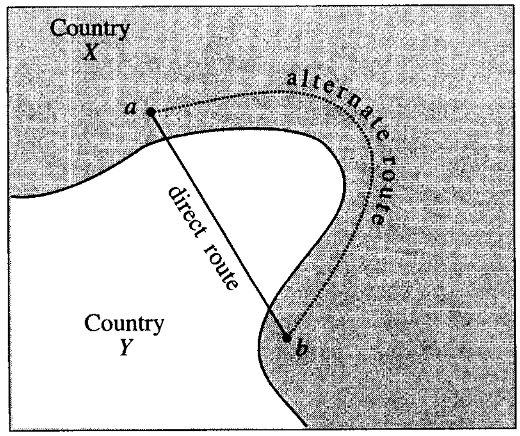


Fig. B.1 Use of complex numbers can reduce the work.

$$z = a + jb \tag{B.1}$$

The numbers a and b (the abscissa and the ordinate) of z are the **real part** and the **imaginary part**, respectively, of z. They are also expressed as

$$\text{Re } z = a$$

$$\text{Im } z = b$$

Note that in this plane all real numbers lie on the horizontal axis, and all imaginary numbers lie on the vertical axis.

Complex numbers may also be expressed in terms of polar coordinates. If (r, θ) are the polar coordinates of a point $z = a + jb$ (see Fig. B.2), then

$$a = r \cos \theta$$

$$b = r \sin \theta$$

and

$$\begin{aligned} z = a + jb &= r \cos \theta + jr \sin \theta \\ &= r(\cos \theta + j \sin \theta) \end{aligned} \tag{B.2}$$

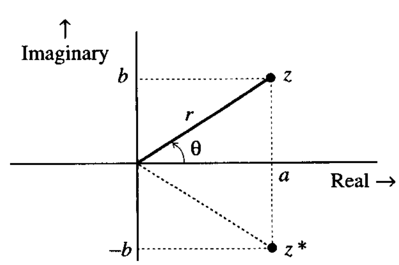


Fig. B.2 Representation of a number in the complex plane.

The **Euler formula** states that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

To prove the Euler formula, we expand $e^{j\theta}$, $\cos\theta$, and $\sin\theta$ using a Maclaurin series

$$\begin{aligned} e^{j\theta} &= 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \frac{(j\theta)^6}{6!} + \cdots \\ &= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \cdots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!}\cdots \\ \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \end{aligned}$$

Hence, it follows that

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{B.3}$$

Using (B.3) in (B.2) yields

$$\begin{aligned} z &= a + jb \\ &= re^{j\theta} \end{aligned} \tag{B.4}$$

Thus, a complex number can be expressed in Cartesian form $a + jb$ or polar form $re^{j\theta}$ with

$$a = r\cos\theta, \qquad b = r\sin\theta \tag{B.5}$$

and

$$r = \sqrt{a^2 + b^2}, \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right) \tag{B.6}$$

Observe that r is the distance of the point z from the origin. For this reason, r is also called the **magnitude** (or **absolute value**) of z and is denoted by $|z|$. Similarly θ is called the angle of z and is denoted by $\angle z$. Therefore

$$|z| = r, \qquad \angle z = \theta$$

and

$$z = |z|e^{j\angle z} \tag{B.7}$$

Also

$$\frac{1}{z} = \frac{1}{re^{j\theta}} = \frac{1}{r}e^{-j\theta} = \frac{1}{|z|}e^{-j\angle z} \tag{B.8}$$

Conjugate of a Complex Number

We define z^*, the **conjugate** of $z = a + jb$, as

$$z^* = a - jb = re^{-j\theta} \tag{B.9a}$$

$$= |z|e^{-j\angle z} \tag{B.9b}$$

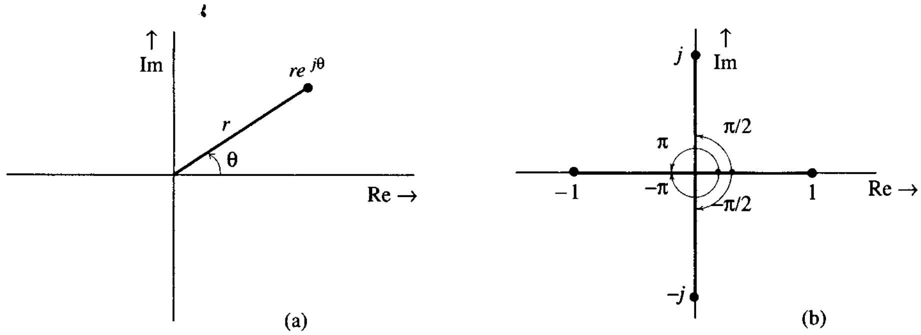


Fig. B.3 Understanding some useful identities in terms of $re^{j\theta}$.

The graphical representation of a number z and its conjugate z^* is depicted in Fig. B.2. Observe that z^* is a mirror image of z about the horizontal axis. *To find the conjugate of any number, we need only to replace j by $-j$ in that number* (which is the same as changing the sign of its angle).

The sum of a complex number and its conjugate is a real number equal to twice the real part of the number:

$$z + z^* = (a + jb) + (a - jb) = 2a = 2\,\mathrm{Re}\,z \tag{B.10a}$$

The product of a complex number z and its conjugate is a real number $|z|^2$, the square of the magnitude of the number:

$$zz^* = (a + jb)(a - jb) = a^2 + b^2 = |z|^2 \tag{B.10b}$$

Understanding Some Useful Identities

In a complex plane, $re^{j\theta}$ represents a point at a distance r from the origin and at an angle θ with the horizontal axis, as shown in Fig. B.3a. For example, the number -1 is at a unit distance from the origin and has an angle π or $-\pi$ (in fact, any odd multiple of $\pm\pi$), as seen from Fig. B.3b. Therefore,

$$1e^{\pm j\pi} = -1$$

In fact,

$$e^{\pm jn\pi} = -1 \qquad n \text{ odd integer} \tag{B.11}$$

The number 1, on the other hand, is also at a unit distance from the origin, but has an angle 2π (in fact, $\pm 2n\pi$ for any integral value of n). Therefore,

$$e^{\pm j2n\pi} = 1 \qquad n \text{ integer} \tag{B.12}$$

The number j is at unit distance from the origin and its angle is $\pi/2$ (see Fig. B.3b). Therefore,

$$e^{j\pi/2} = j$$

Similarly,

$$e^{-j\pi/2} = -j$$

Thus

$$e^{\pm j\pi/2} = \pm j \tag{B.13a}$$

In fact,

$$e^{\pm jn\pi/2} = \pm j \qquad n = 1,\ 5,\ 9,\ 13, \cdots \tag{B.13b}$$

and

$$e^{\pm jn\pi/2} = \mp j \qquad n = 3,\ 7,\ 11,\ 15, \cdots \tag{B.13c}$$

These results are summarized in Table B.1.

TABLE B.1

r	θ	$re^{j\theta}$	
1	0	$e^{j0} = 1$	
1	$\pm\pi$	$e^{\pm j\pi} = -1$	
1	$\pm n\pi$	$e^{\pm jn\pi} = -1$	n odd integer
1	$\pm 2\pi$	$e^{\pm j2\pi} = 1$	
1	$\pm 2n\pi$	$e^{\pm j2n\pi} = 1$	n integer
1	$\pm\pi/2$	$e^{\pm j\pi/2} = \pm j$	
1	$\pm n\pi/2$	$e^{\pm jn\pi/2} = \pm j$	$n = 1, 5, 9, 13, \ldots$
1	$\pm n\pi/2$	$e^{\pm jn\pi/2} = \mp j$	$n = 3, 7, 11, 15, \ldots$

This discussion shows the usefulness of the graphic picture of $re^{j\theta}$. This picture is also helpful in several other applications. For example, to determine the limit of $e^{(\alpha+j\omega)t}$ as $t \to \infty$, we note that

$$e^{(\alpha+j\omega)t} = e^{\alpha t}e^{j\omega t}$$

Now the magnitude of $e^{j\omega t}$ is unity regardless of the value of ω or t because $e^{j\omega t} = re^{j\theta}$ with $r = 1$. Therefore, $e^{\alpha t}$ determines the behavior of $e^{(\alpha+j\omega)t}$ as $t \to \infty$ and

$$\lim_{t\to\infty} e^{(\alpha+j\omega)t} = \lim_{t\to\infty} e^{\alpha t}e^{j\omega t} = \begin{cases} 0 & \alpha < 0 \\ \infty & \alpha > 0 \end{cases} \tag{B.14}$$

In future discussions you will find it very useful to remember $re^{j\theta}$ as a number at a distance r from the origin and at an angle θ with the horizontal axis of the complex plane.

A Warning About Using Electronic Calculators in Computing Angles

From the Cartesian form $a + jb$ we can readily compute the polar form $re^{j\theta}$ [see Eq. (B.6)]. Electronic calculators provide ready conversion of rectangular into polar and vice versa. However, if a calculator computes an angle of a complex number using an inverse trigonometric function $\theta = \tan^{-1}(b/a)$, proper attention must be paid to the quadrant in which the number is located. For instance, θ corresponding to the number $-2 - j3$ is $\tan^{-1}(\frac{-3}{-2})$. This result is not the same as $\tan^{-1}(\frac{3}{2})$. The former is $-123.7°$, whereas the latter is $56.3°$. An electronic calculator cannot make this distinction and can give a correct answer only for angles in the first and

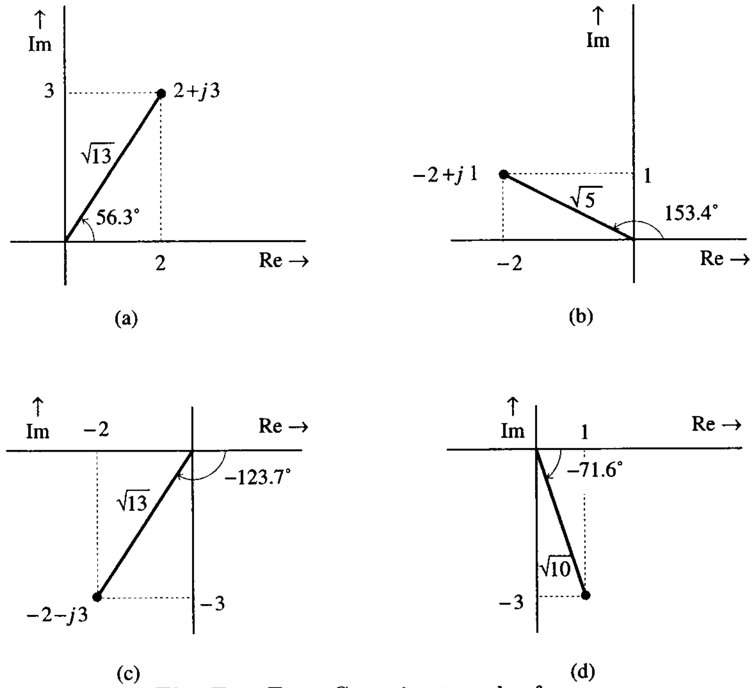


Fig. B.4 From Cartesian to polar form.

fourth quadrants. It will read $\tan^{-1}(\frac{-3}{-2})$ as $\tan^{-1}(\frac{3}{2})$, which is clearly wrong. In computing inverse trigonometric functions, if the angle appears in the second or third quadrant, the answer of the calculator is off by 180°. The correct answer is obtained by adding or subtracting 180° to the value found with the calculator (either adding or subtracting yields the correct answer). For this reason it is advisable to draw the point in the complex plane and determine the quadrant in which the point lies. This issue will be clarified by the following examples.

■ Example B.1

Express the following numbers in polar form:
(a) $2+j3$ **(b)** $-2+j1$ **(c)** $-2-j3$ **(d)** $1-j3$

(a)

$$|z| = \sqrt{2^2+3^2} = \sqrt{13} \qquad \angle z = \tan^{-1}\left(\tfrac{3}{2}\right) = 56.3^\circ$$

In this case the number is in the first quadrant, and a calculator will give the correct value of 56.3°. Therefore, (see Fig. B.4a)

$$2+j3 = \sqrt{13}\,e^{j56.3^\circ}$$

(b)

$$|z| = \sqrt{(-2)^2+1^2} = \sqrt{5} \qquad \angle z = \tan^{-1}\left(\tfrac{1}{-2}\right) = 153.4^\circ$$

In this case the angle is in the second quadrant (see Fig. B.4b), and therefore the answer given by the calculator ($\tan^{-1}(\frac{1}{-2}) = -26.6^\circ$) is off by 180°. The correct answer is $(-26.6 \pm 180)^\circ = 153.4^\circ$ or -206.6°. Both values are correct because they represent the same angle. As a matter of convenience, we choose an angle whose numerical value is less than 180°, which in this case is 153.4°. Therefore,

$$-2+j1 = \sqrt{5}e^{j153.4^\circ}$$

(c)

$$|z| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13} \qquad \angle z = \tan^{-1}(\tfrac{-3}{-2}) = -123.7^\circ$$

In this case the angle appears in the third quadrant (see Fig. B.4c), and therefore the answer obtained by the calculator ($\tan^{-1}(\frac{-3}{-2}) = 56.3^\circ$) is off by 180°. The correct answer is $(56.3 \pm 180)^\circ = 236.3^\circ$ or -123.7°. As a matter of convenience, we choose the latter and (see Fig. B.4c)

$$-2 - j3 = \sqrt{13}e^{-j123.7^\circ}$$

(d)

$$|z| = \sqrt{1^2 + (-3)^2} = \sqrt{10} \qquad \angle z = \tan^{-1}(\tfrac{-3}{1}) = -71.6^\circ$$

In this case the angle appears in the fourth quadrant (see Fig. B.4d), and therefore the answer given by the calculator ($\tan^{-1}(\frac{-3}{1}) = -71.6^\circ$) is correct (see Fig. B.4d).

$$1 - j3 = \sqrt{10}e^{-j71.6^\circ} \quad ■$$

⊙ **Computer Example CB.1**

Express the following numbers in polar form: **(a)** $2 + j3$ **(b)** $-2 + j1$

MATLAB function *cart2pol(a,b)* can be used to convert the complex number $a + jb$ to its polar form.

(a)

```
[Zangle_in_rad,Zmag]=cart2pol(2,3)
Zangle_in_rad = 0.9828
Zmag =3.6056
Zangle_in_deg=Zangle_in_rad*(180/pi)
Zangle_in_deg=56.31
```

Therefore

$$z = 2 + j3 = 3.6056e^{j56.31^\circ}$$

(b)

```
[Zangle_in_rad,Zmag]=cart2pol(-2,1)
Zangle_in_rad = 2.6779
Zmag =2.2361
Zangle_in_deg=Zangle_in_rad*(180/pi)
Zangle_in_deg=153.4349
```

Therefore

$$z = -2 + j1 = 2.2361e^{j153.4349^\circ}$$

Note that MATLAB automatically takes care of the quadrant in which the complex number lies. ⊙

■ **Example B.2**

Represent the following numbers in the complex plane and express them in Cartesian form: **(a)** $2e^{j\pi/3}$ **(b)** $4e^{-j3\pi/4}$ **(c)** $2e^{j\pi/2}$ **(d)** $3e^{-j3\pi}$ **(e)** $2e^{j4\pi}$ **(f)** $2e^{-j4\pi}$.

(a) $2e^{j\pi/3} = 2\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right) = 1 + j\sqrt{3}$ (see Fig. B.5a)

(b) $4e^{-j3\pi/4} = 4\left(\cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}\right) = -2\sqrt{2} - j2\sqrt{2}$ (see Fig. B.5b)

(c) $2e^{j\pi/2} = 2\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right) = 2(0 + j1) = j2$ (see Fig. B.5c)

(d) $3e^{-j3\pi} = 3(\cos 3\pi - j\sin 3\pi) = 3(-1 + j0) = -3$ (see Fig. B.5d)

(e) $2e^{j4\pi} = 2(\cos 4\pi + j\sin 4\pi) = 2(1 + j0) = 2$ (see Fig. B.5e)

(f) $2e^{-j4\pi} = 2(\cos 4\pi - j\sin 4\pi) = 2(1 - j0) = 2$ (see Fig. B.5f) ■

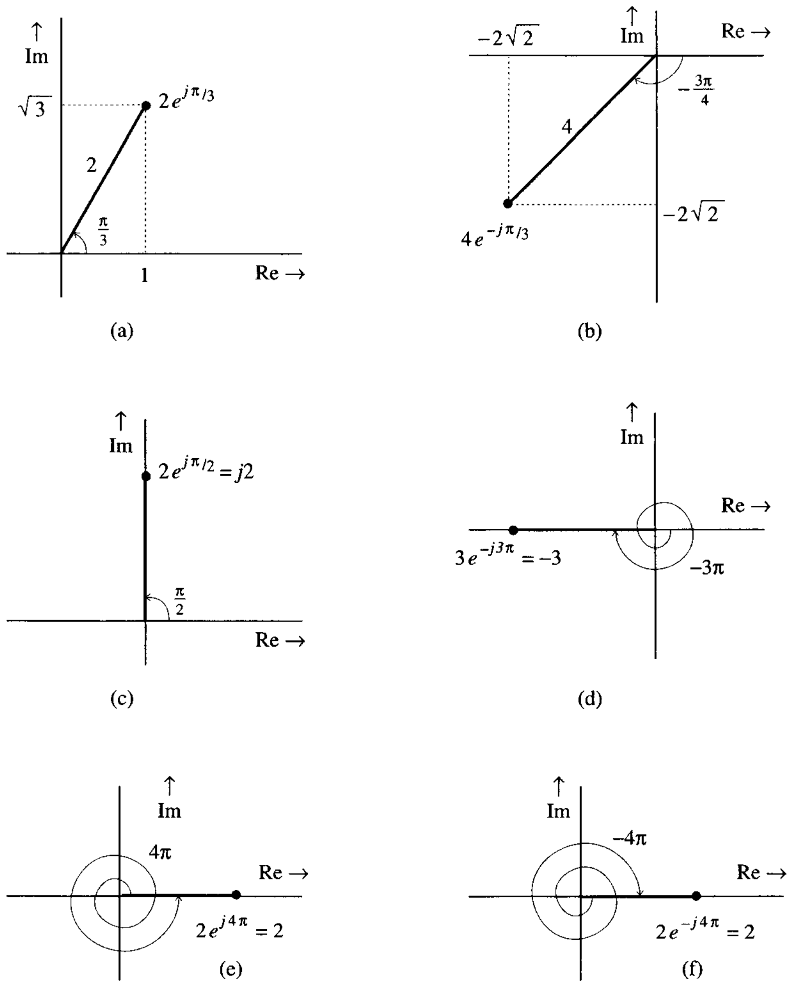


Fig. B.5 From polar to Cartesian form.

⊙ **Computer Example CB.2**

Represent $4e^{-j\frac{3\pi}{4}}$ in Cartesian form.

MATLAB function pol2cart(θ, r) converts the complex number $re^{j\theta}$ to Cartesian form.

```
[Zreal,Zimag]=pol2cart(-3*pi/4,4)
Zreal=-2.8284
Zimag=-2.8284
```

Therefore

$$4e^{-j\frac{3\pi}{4}} = -2.8284 - j2.8284$$ ⊙

Arithmetical Operations, Powers, and Roots of Complex Numbers

To perform addition and subtraction, complex numbers should be expressed in Cartesian form. Thus, if

$$z_1 = 3 + j4 = 5e^{j53.1^\circ}$$

and

$$z_2 = 2 + j3 = \sqrt{13}e^{j56.3^\circ}$$

then

$$z_1 + z_2 = (3 + j4) + (2 + j3) = 5 + j7$$

If z_1 and z_2 are given in polar form, we would need to convert them into Cartesian form for the purpose of adding (or subtracting). Multiplication and division, however, can be carried out in either Cartesian or polar form, although the latter proves to be much more convenient. This is because if z_1 and z_2 are expressed in polar form as

$$z_1 = r_1 e^{j\theta_1} \qquad \text{and} \qquad z_2 = r_2 e^{j\theta_2}$$

then

$$z_1 z_2 = \left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1+\theta_2)} \tag{B.15a}$$

and

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)} \tag{B.15b}$$

Moreover,

$$z^n = \left(re^{j\theta}\right)^n = r^n e^{jn\theta} \tag{B.15c}$$

and

$$z^{1/n} = \left(re^{j\theta}\right)^{1/n} = r^{1/n} e^{j\theta/n} \tag{B.15d}$$

This shows that the operations of multiplication, division, powers, and roots can be carried out with remarkable ease when the numbers are in polar form.

■ **Example B.3**

Determine $z_1 z_2$ and z_1/z_2 for the numbers

$$z_1 = 3 + j4 = 5e^{j53.1^\circ}$$

$$z_2 = 2 + j3 = \sqrt{13}e^{j56.3^\circ}$$

We shall solve this problem in both polar and Cartesian forms.

Multiplication: Cartesian Form

$$z_1 z_2 = (3 + j4)(2 + j3) = (6 - 12) + j(8 + 9) = -6 + j17$$

Multiplication: Polar Form

$$z_1 z_2 = \left(5e^{j53.1^\circ}\right)\left(\sqrt{13}e^{j56.3^\circ}\right) = 5\sqrt{13}e^{j109.4^\circ}$$

Division: Cartesian Form

$$\frac{z_1}{z_2} = \frac{3 + j4}{2 + j3}$$

In order to eliminate the complex number in the denominator, we multiply both the numerator and the denominator of the right-hand side by $2 - j3$, the denominator's conjugate. This yields

$$\frac{z_1}{z_2} = \frac{(3+j4)(2-j3)}{(2+j3)(2-j3)} = \frac{18-j1}{2^2+3^2} = \frac{18-j1}{13} = \frac{18}{13} - j\frac{1}{13}$$

Division: Polar Form

$$\frac{z_1}{z_2} = \frac{5e^{j53.1^\circ}}{\sqrt{13}e^{j56.3^\circ}} = \frac{5}{\sqrt{13}}e^{j(53.1^\circ - 56.3^\circ)} = \frac{5}{\sqrt{13}}e^{-j3.2^\circ} \quad \blacksquare$$

It is clear from this example that multiplication and division are easier to accomplish in polar form than in Cartesian form.

■ **Example B.4**

For $z_1 = 2e^{j\pi/4}$ and $z_2 = 8e^{j\pi/3}$, find **(a)** $2z_1 - z_2$ **(b)** $\frac{1}{z_1}$ **(c)** $\frac{z_1}{z_2^2}$ **(d)** $\sqrt[3]{z_2}$

(a) Since subtraction cannot be performed directly in polar form, we convert z_1 and z_2 to Cartesian form:

$$z_1 = 2e^{j\pi/4} = 2\left(\cos\tfrac{\pi}{4} + j\sin\tfrac{\pi}{4}\right) = \sqrt{2} + j\sqrt{2}$$

$$z_2 = 8e^{j\pi/3} = 8\left(\cos\tfrac{\pi}{3} + j\sin\tfrac{\pi}{3}\right) = 4 + j4\sqrt{3}$$

Therefore,

$$\begin{aligned} 2z_1 - z_2 &= 2(\sqrt{2} + j\sqrt{2}) - (4 + j4\sqrt{3}) \\ &= (2\sqrt{2} - 4) + j(2\sqrt{2} - 4\sqrt{3}) \\ &= -1.17 - j4.1 \end{aligned}$$

(b)

$$\frac{1}{z_1} = \frac{1}{2e^{j\pi/4}} = \frac{1}{2}e^{-j\pi/4}$$

(c)

$$\frac{z_1}{z_2^2} = \frac{2e^{j\pi/4}}{(8e^{j\pi/3})^2} = \frac{2e^{j\pi/4}}{64e^{j2\pi/3}} = \frac{1}{32}e^{j(\frac{\pi}{4} - \frac{2\pi}{3})} = \frac{1}{32}e^{-j\frac{5\pi}{12}}$$

(d)

$$\sqrt[3]{z_2} = z_2^{1/3} = \left(8e^{j\pi/3}\right)^{\frac{1}{3}} = 8^{\frac{1}{3}}\left(e^{j\pi/3}\right)^{1/3} = 2e^{j\pi/9} \quad \blacksquare$$

⊙ **Computer Example CB.3**

Determine $z_1 z_2$ and z_1/z_2 if $z_1 = 3 + j4$ and $z_2 = 2 + j3$

Multiplication and division: Cartesian Form

```
z1=3+j*4; z2=2+j*3;
z1z2=z1*z2
z1z2=-6.000+17.0000i
z1_over_z2=z1/z2
z1_over_z2=1.3486-0.0769i
```

Therefore

$$(3+j4)(2+j3) = -6 + j17 \quad \text{and} \quad (3+j4)/(2+j3) = 1.3486 - 0.0769 \quad \odot$$

■ **Example B.5**

Consider $F(\omega)$, a complex function of a real variable ω:

$$F(\omega) = \frac{2 + j\omega}{3 + j4\omega} \tag{B.16a}$$

(a) Express $F(\omega)$ in Cartesian form, and find its real and imaginary parts. **(b)** Express $F(\omega)$ in polar form, and find its magnitude $|F(\omega)|$ and angle $\angle F(\omega)$.

(a) To obtain the real and imaginary parts of $F(\omega)$, we must eliminate imaginary terms in the denominator of $F(\omega)$. This is readily done by multiplying both the numerator and denominator of $F(\omega)$ by $3 - j4\omega$, the conjugate of the denominator $3 + j4\omega$ so that

$$F(\omega) = \frac{(2 + j\omega)(3 - j4\omega)}{(3 + j4\omega)(3 - j4\omega)} = \frac{(6 + 4\omega^2) - j5\omega}{9 + 16\omega^2} = \frac{6 + 4\omega^2}{9 + 16\omega^2} - j\frac{5\omega}{9 + \omega^2} \tag{B.16b}$$

This is the Cartesian form of $F(\omega)$. Clearly the real and imaginary parts $F_r(\omega)$ and $F_i(\omega)$ are given by

$$F_r(\omega) = \frac{6 + 4\omega^2}{9 + 16\omega^2}, \qquad F_i(\omega) = \frac{-5\omega}{9 + 16\omega^2}$$

(b)

$$\begin{aligned} F(\omega) = \frac{2 + j\omega}{3 + j4\omega} &= \frac{\sqrt{4 + \omega^2}\, e^{j\tan^{-1}\left(\frac{\omega}{2}\right)}}{\sqrt{9 + 16\omega^2}\, e^{j\tan^{-1}\left(\frac{4\omega}{3}\right)}} \\ &= \sqrt{\frac{4 + \omega^2}{9 + 16\omega^2}}\, e^{j\left[\tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{4\omega}{3}\right)\right]} \end{aligned} \tag{B.16c}$$

This is the polar representation of $F(\omega)$. Observe that

$$|F(\omega)| = \sqrt{\frac{4 + \omega^2}{9 + 16\omega^2}}, \qquad \angle F(\omega) = \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{4\omega}{3}\right) \tag{B.17}$$

■

B.2 Sinusoids

Consider the sinusoid

$$f(t) = C\cos\left(2\pi\mathcal{F}_0 t + \theta\right) \tag{B.18}$$

We know that

$$\cos\varphi = \cos\left(\varphi + 2n\pi\right) \qquad n = 0, \pm1, \pm2, \pm3, \cdots$$

Therefore, $\cos\varphi$ repeats itself for every change of 2π in the angle φ. For the sinusoid in Eq. (B.18), the angle $2\pi\mathcal{F}_0 t + \theta$ changes by 2π when t changes by $1/\mathcal{F}_0$. Clearly, this sinusoid repeats every $1/\mathcal{F}_0$ seconds. As a result, there are $\mathcal{F}_0$ repetitions per second. This is the **frequency** of the sinusoid, and the repetition interval T_0 given by

$$T_0 = \frac{1}{\mathcal{F}_0} \tag{B.19}$$

is the **period**. For the sinusoid in Eq. (B.18), C is the **amplitude**, $\mathcal{F}_0$ is the **frequency** (in **Hertz**), and θ is the phase. Let us consider two special cases of this sinusoid when $\theta = 0$ and $\theta = -\pi/2$ as follows:

(a) $f(t) = C \cos 2\pi\mathcal{F}_0 t \qquad (\theta = 0)$

(b) $f(t) = C \cos\left(2\pi\mathcal{F}_0 t - \frac{\pi}{2}\right) = C \sin 2\pi\mathcal{F}_0 t \qquad (\theta = -\pi/2)$

The angle or phase can be expressed in units of degrees or radians. Although the radian is the proper unit, in this book we shall often use the degree unit because students generally have a better feel for the relative magnitudes of angles when expressed in degrees rather than in radians. For example, we relate better to the angle 24° than to 0.419 radians. Remember, however, when in doubt, use the radian unit and, above all, be consistent. In other words, in a given problem or an expression do not mix the two units.

It is convenient to use the variable ω_0 (*radian frequency*) to express $2\pi\mathcal{F}_0$:

$$\omega_0 = 2\pi\mathcal{F}_0 \tag{B.20}$$

With this notation, the sinusoid in Eq. (B.18) can be expressed as

$$f(t) = C \cos(\omega_0 t + \theta)$$

in which the period T_0 is given by [see Eqs. (B.19) and (B.20)]

$$T_0 = \frac{1}{\omega_0/2\pi} = \frac{2\pi}{\omega_0} \tag{B.21a}$$

and

$$\omega_0 = \frac{2\pi}{T_0} \tag{B.21b}$$

In future discussions, we shall often refer to ω_0 as the frequency of the signal $\cos(\omega_0 t + \theta)$, but it should be clearly understood that the frequency of this sinusoid is $\mathcal{F}_0$ Hz ($\mathcal{F}_0 = \omega_0/2\pi$), and ω_0 is actually the **radian frequency**.

The signals $C \cos \omega_0 t$ and $C \sin \omega_0 t$ are illustrated in Figs. B.6a and B.6b respectively. A general sinusoid $C \cos(\omega_0 t + \theta)$ can be readily sketched by shifting the signal $C \cos \omega_0 t$ in Fig. B.6a by the appropriate amount. Consider, for example,

$$f(t) = C \cos(\omega_0 t - 60°)$$

This signal can be obtained by shifting (delaying) the signal $C \cos \omega_0 t$ (Fig. B.6a) to the right by a phase (angle) of 60°. We know that a sinusoid undergoes a 360° change of phase (or angle) in one cycle. A quarter-cycle segment corresponds to a 90° change of angle. Therefore, an angle of 60° corresponds to two-thirds of a quarter-cycle segment. We therefore shift (delay) the signal in Fig. B.6a by two-thirds of a quarter-cycle segment to obtain $C \cos(\omega_0 t - 60°)$, as shown in Fig. B.6c.

Observe that if we delay $C \cos \omega_0 t$ in Fig. B.6a by a quarter-cycle (angle of 90° or $\pi/2$ radians), we obtain the signal $C \sin \omega_0 t$, depicted in Fig. B.6b. This verifies the well-known trigonometric identity

$$C \cos\left(\omega_0 t - \frac{\pi}{2}\right) = C \sin \omega_0 t \tag{B.22a}$$

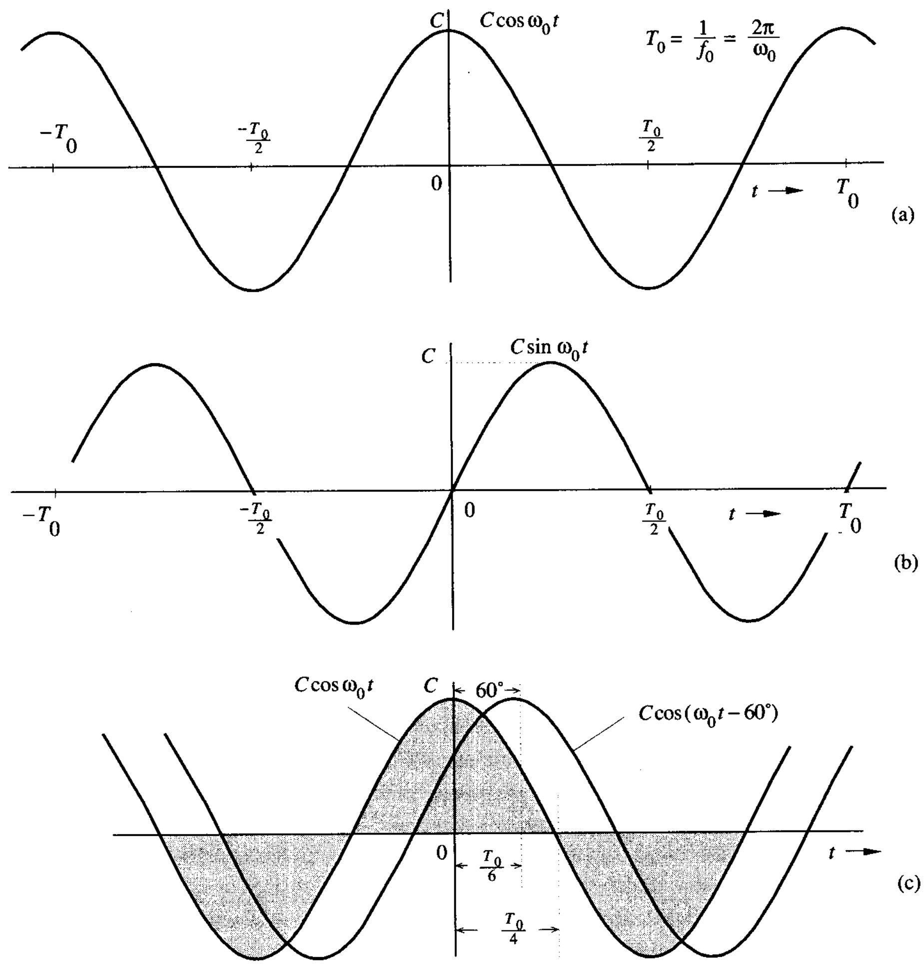


Fig. B.6 Sketching a sinusoid.

Alternatively, if we advance $C \sin \omega_0 t$ by a quarter-cycle, we obtain $C \cos \omega_0 t$. Therefore,

$$C \sin \left(\omega_0 t + \frac{\pi}{2}\right) = C \cos \omega_0 t \tag{B.22b}$$

This observation means $\sin \omega_0 t$ lags $\cos \omega_0 t$ by $90°(\pi/2$ radians$)$, or $\cos \omega_0 t$ leads $\sin \omega_0 t$ by $90°$.

B.2-1 Addition of Sinusoids

Two sinusoids having the same frequency but different phases add to form a single sinusoid of the same frequency. This fact is readily seen from the well-known trigonometric identity

$$\begin{aligned} C \cos (\omega_0 t + \theta) &= C \cos \theta \cos \omega_0 t - C \sin \theta \sin \omega_0 t \\ &= a \cos \omega_0 t + b \sin \omega_0 t \end{aligned} \tag{B.23a}$$

in which

$$a = C \cos \theta, \qquad b = -C \sin \theta$$

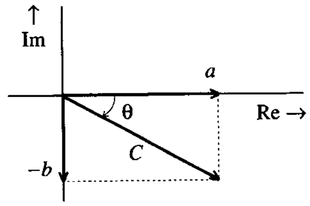


Fig. B.7 Phasor addition of sinusoids.

Therefore,

$$C = \sqrt{a^2 + b^2} \tag{B.23b}$$

$$\theta = \tan^{-1}\left(\frac{-b}{a}\right) \tag{B.23c}$$

Equations (B.23b) and (B.23c) show that C and θ are the magnitude and angle, respectively, of a complex number $a - jb$. In other words, $a - jb = Ce^{j\theta}$. Hence, to find C and θ, we convert $a - jb$ to polar form and the magnitude and the angle of the resulting polar number are C and θ, respectively.

To summarize,

$$a \cos \omega_0 t + b \sin \omega_0 t = C \cos (\omega_0 t + \theta)$$

in which C and θ are given by Eqs. (B.23b) and (B.23c), respectively. These happen to be the magnitude and angle, respectively, of $a - jb$.

The process of adding two sinusoids with the same frequency can be clarified by using **phasors** to represent sinusoids. We represent the sinusoid $C \cos(\omega_0 t + \theta)$ by a phasor of length C at an angle θ with the horizontal axis. Clearly, the sinusoid $a \cos \omega_0 t$ is represented by a horizontal phasor of length a $(\theta = 0)$, while $b \sin \omega_0 t = b \cos(\omega_0 t - \frac{\pi}{2})$ is represented by a vertical phasor of length b at an angle $-\pi/2$ with the horizontal (Fig. B.7). Adding these two phasors results in a phasor of length C at an angle θ, as depicted in Fig. B.7. From this figure, we verify the values of C and θ found in Eqs. (B.23b) and (B.23c), respectively.

Proper care should be exercised in computing θ. Recall that $\tan^{-1}(\frac{-b}{a}) \neq \tan^{-1}(\frac{b}{-a})$. Similarly, $\tan^{-1}(\frac{-b}{-a}) \neq \tan^{-1}(\frac{b}{a})$. Electronic calculators cannot make this distinction. When calculating such an angle, it is advisable to note the quadrant where the angle lies and not to rely exclusively on an electronic calculator. A foolproof method is to convert the complex number $a - jb$ to polar form. The magnitude of the resulting polar number is C and the angle is θ. The following examples clarify this point.

■ **Example B.6**

In the following cases, express $f(t)$ as a single sinusoid:

(a) $f(t) = \cos \omega_0 t - \sqrt{3} \sin \omega_0 t$

(b) $f(t) = -3 \cos \omega_0 t + 4 \sin \omega_0 t$

(a) In this case, $a = 1, b = -\sqrt{3}$, and from Eqs. (B.23)

$$C = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60^\circ$$

Therefore,

$$f(t) = 2\cos\,(\omega_0 t + 60^\circ)$$

We can verify this result by drawing phasors corresponding to the two sinusoids. The sinusoid $\cos\,\omega_0 t$ is represented by a phasor of unit length at a zero angle with the horizontal. The phasor $\sin\,\omega_0 t$ is represented by a unit phasor at an angle of -90° with the horizontal. Therefore, $-\sqrt{3}\sin\,\omega_0 t$ is represented by a phasor of length $\sqrt{3}$ at 90° with the horizontal, as depicted in Fig. B.8a. The two phasors added yield a phasor of length 2 at 60° with the horizontal (also shown in Fig. B.8a). Therefore,

$$f(t) = 2\cos\,(\omega_0 t + 60^\circ)$$

Alternately, we note that $a - jb = 1 + j\sqrt{3} = 2e^{j\pi/3}$. Hence, $C = 2$ and $\theta = \pi/3$.

Observe that a phase shift of $\pm\pi$ amounts to multiplication by -1. Therefore, $f(t)$ can also be expressed alternatively as

$$\begin{aligned} f(t) &= -2\cos\,(\omega_0 t + 60^\circ \pm 180^\circ) \\ &= -2\cos\,(\omega_0 t - 120^\circ) \\ &= -2\cos\,(\omega_0 t + 240^\circ) \end{aligned}$$

In practice, an expression with an angle whose numerical value is less than 180° is preferred.

(b) In this case, $a = -3, b = 4$, and from Eqs. (B.23)

$$C = \sqrt{(-3)^2 + 4^2} = 5$$

$$\theta = \tan^{-1}\left(\tfrac{-4}{-3}\right) = -126.9^\circ$$

Observe that

$$\tan^{-1}\left(\tfrac{-4}{-3}\right) \neq \tan^{-1}\left(\tfrac{4}{3}\right) = 53.1^\circ$$

Therefore,

$$f(t) = 5\cos\,(\omega_0 t - 126.9^\circ)$$

This result is readily verified in the phasor diagram in Fig. B.8b. Alternately, $a - jb = -3 - j4 = 5e^{-j126.9^\circ}$. Hence, $C = 5$ and $\theta = -126.9^\circ$. ■

⊙ Computer Example CB.4

Express $f(t) = -3\cos\,\omega_0 t + 4\sin\,\omega_0 t$ as a single sinusoid.

Recall that $a\cos\,\omega_0 t + b\sin\,\omega_0 t = C\cos\,[\omega_0 t + \tan^{-1}(-b/a)]$. Hence, the amplitude C and the angle θ of the resulting sinusoid are the magnitude and angle of a complex number $a - jb$. We use the 'cart2pol' function to convert it to the polar form to obtain C and θ.

```
a=-3;b=4;
[theta,C]=cart2pol(a,-b);
Theta_deg=(180/pi)*theta;
C,Theta_deg
C=5
Theta_deg=-126.8699
```

Therefore

$$-3\cos\,\omega_0 t + 4\sin\,\omega_0 t = 5\cos\,(\omega_0 t - 126.8699^\circ) \quad \odot$$

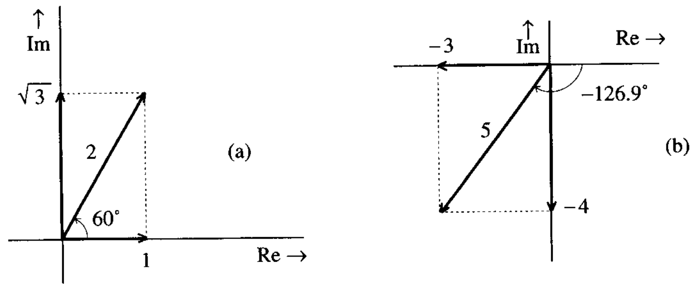


Fig. B.8 Phasor addition of sinusoids in Example B.6.

We can also perform the reverse operation, expressing

$$f(t) = C \cos(\omega_0 t + \theta)$$

in terms of cos $\omega_0 t$ and sin $\omega_0 t$ using the trigonometric identity

$$C \cos(\omega_0 t + \theta) = C \cos\theta \cos\omega_0 t - C \sin\theta \sin\omega_0 t$$

For example,

$$10 \cos(\omega_0 t - 60°) = 5 \cos\omega_0 t + 5\sqrt{3} \sin\omega_0 t$$

Sinusoids in Terms of Exponentials: Euler's Formula

Sinusoids can be expressed in terms of exponentials using Euler's formula [see Eq. (B.3)]

$$\cos\varphi = \frac{1}{2}\left(e^{j\varphi} + e^{-j\varphi}\right) \tag{B.24a}$$

$$\sin\varphi = \frac{1}{2j}\left(e^{j\varphi} - e^{-j\varphi}\right) \tag{B.24b}$$

Inversion of these equations yields

$$e^{j\varphi} = \cos\varphi + j\sin\varphi \tag{B.25a}$$

$$e^{-j\varphi} = \cos\varphi - j\sin\varphi \tag{B.25b}$$

B.3 Sketching Signals

In this section we discuss the sketching of a few useful signals, starting with exponentials.

B.3-1 Monotonic Exponentials

The signal e^{-at} decays monotonically, and the signal e^{at} grows monotonically with t (assuming $a > 0$) as depicted in Fig. B.9. For the sake of simplicity, we shall consider an exponential e^{-at} starting at $t = 0$, as shown in Fig. B.10a.

The signal e^{-at} has a unit value at $t = 0$. At $t = 1/a$, the value drops to $1/e$ (about 37% of its initial value), as illustrated in Fig. B.10a. This time interval over which the exponential reduces by a factor e (that is, drops to about 37% of its value) is known as the **time constant** of the exponential. Therefore, the time

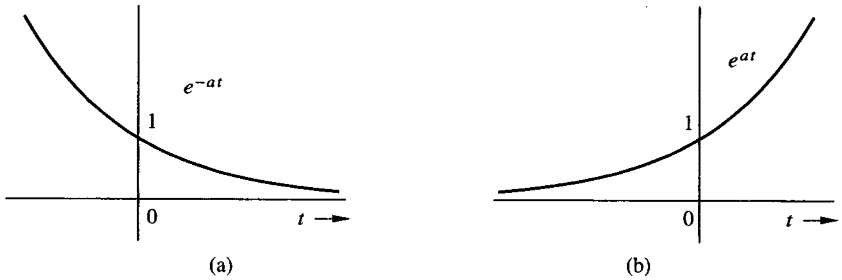


Fig. B.9 Monotonic exponentials.

constant of e^{-at} is $1/a$. Observe that the exponential is reduced to 37% of its initial value over any time interval of duration $1/a$. This can be shown by considering any set of instants t_1 and t_2 separated by one time constant so that

$$t_2 - t_1 = \tfrac{1}{a}$$

Now the ratio of e^{-at_2} to e^{-at_1} is given by

$$\frac{e^{-at_2}}{e^{-at_1}} = e^{-a(t_2-t_1)} = \tfrac{1}{e} \approx 0.37$$

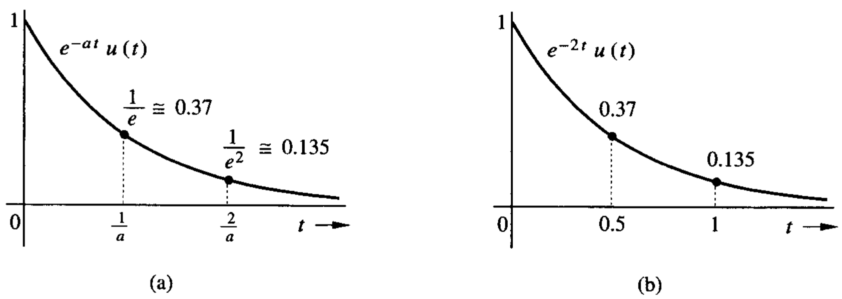


Fig. B.10 (a) Sketching e^{-at} (b) sketching e^{-2t}.

We can use this fact to sketch an exponential quickly. For example, consider

$$f(t) = e^{-2t}$$

The time constant in this case is $1/2$. The value of $f(t)$ at $t = 0$ is 1. At $t = 1/2$ (one time constant) it is $1/e$ (about 0.37). The value of $f(t)$ continues to drop further by the factor $1/e$ (37%) over the next half-second interval (one time constant). Thus $f(t)$ at $t = 1$ is $(1/e)^2$. Continuing in this manner, we see that $f(t) = (1/e)^3$ at $t = 3/2$ and so on. A knowledge of the values of $f(t)$ at $t = 0$, 0.5, 1, and 1.5 allows us to sketch the desired signal† as shown in Fig. B.10b. For a monotonically

†If we wish to refine the sketch further, we could consider intervals of half the time constant over which the signal decays by a factor $1/\sqrt{e}$. Thus, at $t = 0.25$, $f(t) = 1/\sqrt{e}$, and at $t = 0.75$, $f(t) = 1/e\sqrt{e}$, etc.

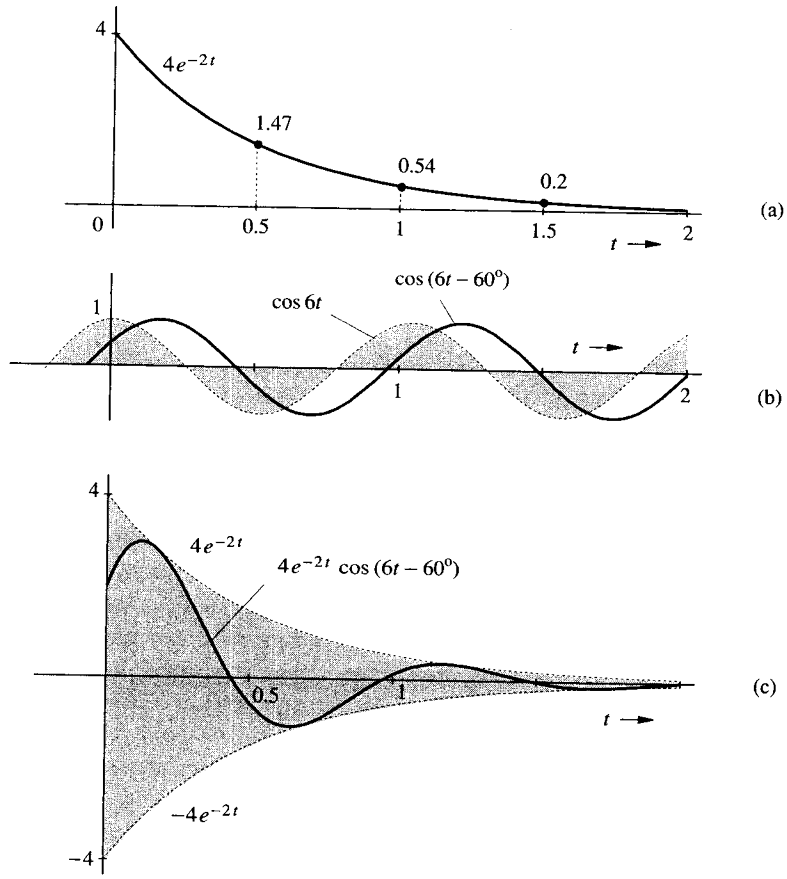


Fig. B.11 Sketching an exponentially varying sinusoid

growing exponential e^{at}, the waveform increases by a factor e over each interval of $1/a$ seconds.

B.3-2 The Exponentially Varying Sinusoid

We now discuss sketching an exponentially varying sinusoid

$$f(t) = Ae^{-at}\cos(\omega_0 t + \theta)$$

Let us consider a specific example

$$f(t) = 4e^{-2t}\cos(6t - 60°) \tag{B.26}$$

We shall sketch $4e^{-2t}$ and $\cos(6t - 60°)$ separately and then multiply them.

(i) Sketching $4e^{-2t}$

This monotonically decaying exponential has a time constant of 1/2 second and an initial value of 4 at $t = 0$. Therefore, its values at $t = 0.5$, 1, 1.5, and 2 are $4/e$, $4/e^2$, $4/e^3$, and $4/e^4$, or about 1.47, 0.54, 0.2, and 0.07 respectively. Using these values as a guide, we sketch $4e^{-2t}$, as illustrated in Fig. B.11a.

(ii) Sketching $\cos(6t - 60°)$

The procedure for sketching $\cos(6t - 60°)$ is discussed in Sec. B.2 (Fig. B.6c). Here the period of the sinusoid is $T_0 = 2\pi/6 \approx 1$, and there is a phase delay of 60°, or two-thirds of a quarter-cycle, which is equivalent to about a $(60/360)(1) \approx 1/6$ second delay (see Fig. B.11b).

(iii) Sketching $4e^{-2t}\cos(6t - 60°)$

We now multiply the waveforms in **(i)** and **(ii)**. The multiplication amounts to forcing the amplitude of the sinusoid $\cos(6t - 60°)$ to decrease exponentially with a time constant of 0.5. The initial amplitude (at $t = 0$) is 4, decreasing to $4/e$ (=1.47) at $t = 0.5$, to $1.47/e$ (=0.54) at $t = 1$, and so on. This is depicted in Fig. B.11c. Note that at the instants where $\cos(6t - 60°)$ has a value of unity (peak amplitude),

$$4e^{-2t}\cos(6t - 60°) = 4e^{-2t} \tag{B.27}$$

Therefore, $4e^{-2t}\cos(6t - 60°)$ touches $4e^{-2t}$ at those instants where the sinusoid $\cos(6t - 60°)$ is at its positive peaks. Clearly $4e^{-2t}$ is an envelope for positive amplitudes of $4e^{-2t}\cos(6t - 60°)$. Similarly, at those instants where the sinusoid $\cos(6t - 60°)$ has a value of -1 (negative peak amplitude),

$$4e^{-2t}\cos(6t - 60°) = -4e^{-2t} \tag{B.28}$$

and $4e^{-2t}\cos(6t - 60°)$ touches $-4e^{-2t}$ at its negative peaks. Therefore, $-4e^{-2t}$ is an envelope for negative amplitudes of $4e^{-2t}\cos(6t - 60°)$. Thus, to sketch $4e^{-2t}\cos(6t - 60°)$, we first draw the envelopes $4e^{-2t}$ and $-4e^{-2t}$ (the mirror image of $4e^{-2t}$ about the horizontal axis), and then sketch the sinusoid $\cos(6t - 60°)$, with these envelopes acting as constraints on the sinusoid's amplitude (see Fig. B.11c).

In general, $Ke^{-at}\cos(\omega_0 t + \theta)$ can be sketched in this manner, with Ke^{-at} and $-Ke^{-at}$ constraining the amplitude of $\cos(\omega_0 t + \theta)$.

B.4 Cramer's Rule

This is a very convenient rule used to solve simultaneous linear equations. Consider a set of n linear simultaneous equations in n unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= y_n \end{aligned} \tag{B.29}$$

These equations can be expressed in matrix form as