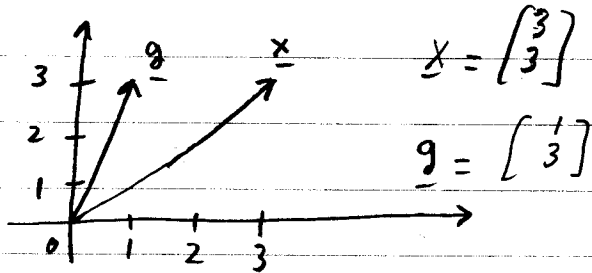


Vectors, Signals, and Fourier series

Vectors:



$\underline{g} = c \underline{x} + \underline{e}$: The "error" vector \underline{e} is a function of c .

What value of c gives the smallest length of \underline{e} ?

"Length" of a vector : $|\underline{x}| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$

$$|\underline{g}| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

"Dot product" or "scalar product" or "inner product":

$$(\underline{x}, \underline{g}) = \underline{x} \cdot \underline{g} = 3 \cdot 1 + 3 \cdot 3 = 12$$

Some properties: Dot product is linear:

$$\text{For scalar } a, (\underline{ax}, \underline{g}) = (\underline{x}, a\underline{g}) = a(\underline{x}, \underline{g})$$

$$(\underline{x}, \underline{g}_1 + \underline{g}_2) = (\underline{x}, \underline{g}_1) + (\underline{x}, \underline{g}_2)$$

Length of \underline{e} as a function of c :

$$|\underline{e}(c)|^2 = \underline{e}(c) \cdot \underline{e}(c) = [\underline{g} - c\underline{x}] \cdot [\underline{g} - c\underline{x}]$$

$$= |\underline{g}|^2 + c^2 |\underline{x}|^2 - 2c \underline{x} \cdot \underline{g}$$

What value of c minimizes $|\underline{e}(c)|^2$?

$$\frac{d}{dc} |\underline{e}(c)|^2 = 0 \quad \Rightarrow \quad c = \frac{\underline{x} \cdot \underline{g}}{|\underline{x}|^2} = \frac{12}{18} = \frac{2}{3}$$

Def: vectors are orthogonal if $\underline{x} \cdot \underline{g} = 0$.

Equivalent condition for minimum $|\underline{e}(c)|^2$:

$c\underline{x}$ is the orthogonal projection of \underline{g} on \underline{x} .

$$\left\{ \begin{array}{l} \underline{e} \cdot \underline{x} = 0 \\ [\underline{g} - c\underline{x}] \cdot \underline{x} = 0 \\ \underline{g} \cdot \underline{x} - c|\underline{x}|^2 = 0 \Rightarrow c = \frac{\underline{g} \cdot \underline{x}}{|\underline{x}|^2} \end{array} \right.$$

Signals:

"Length" of $g(t)$ is $\sqrt{E_g}$, $E_g = \int_{-\infty}^{\infty} g(t)^2 dt$

Inner product of two signals $x(t), g(t)$:

$$(x, g) = \int_{-\infty}^{\infty} x(t)g(t) dt$$

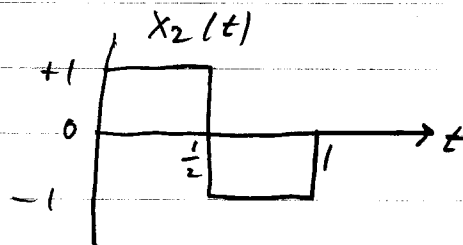
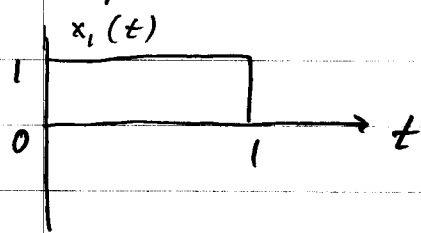
Note $E_g = (g, g)$

How to choose scalar c to approximate a signal $g(t)$ with another signal $x(t)$ so that the "error energy" is minimum?

$$g(t) = c x(t) + e(t)$$

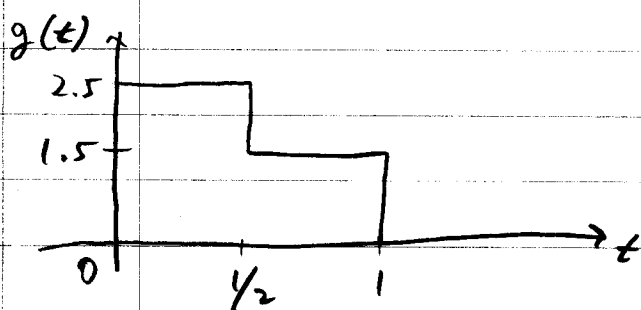
Answer: $c =$

Example: Simple "wavelet"



Are $x_1(t)$ and $x_2(t)$ orthogonal?

$$(x_1, x_2) = \int_0^1 x_1(t) x_2(t) dt =$$



"Best" approximation of $g(t)$ with $x_1(t)$:

$$g(t) = c_1 x_1(t) + e_1(t)$$

$$\Rightarrow c_1 = \frac{\int_0^1 g(t) x_1(t) dt}{\int_0^1 x_1(t)^2 dt} =$$

"Best" approximation of $g(t)$ with $x_2(t)$:

$$g(t) = c_2 x_2(t) + e_2(t)$$

$$\Rightarrow c_2 = \frac{\int_0^1 g(t) x_2(t) dt}{\int_0^1 x_2(t)^2 dt} =$$

Note for this case,

$$g(t) = \boxed{} x_1(t) + \boxed{} x_2(t) \text{ is exact}$$

(see example 2.5 in text)

(no error).

Exponential Fourier series:

Energy and inner product of complex signals:

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = (g, g)$$

$$(g, x) = \int_{-\infty}^{\infty} g(t) x(t)^* dt$$

Fourier series for periodic $g(t)$ with period T_0 :

We use "basis functions" that are complex exponentials:

$$x_n(t) = e^{jn\omega_0 t}, \quad n = 0, \pm 1, \pm 2, \dots, \quad \omega_0 = \frac{2\pi}{T_0}$$

These are orthogonal over one period T_0 :

$$(x_n, x_m) = \int_{T_0} x_n(t) x_m(t)^* dt =$$

$$= \int_{T_0} e^{jn\omega_0 t} e^{-jm\omega_0 t} dt = \begin{cases} 0, & n \neq m \\ T_0, & n = m \end{cases}$$

The set $\{e^{jn\omega_0 t}, n = 0, \pm 1, \pm 2, \dots\}$ is also "complete".

How do we choose $D_n, n = 0, \pm 1, \pm 2, \dots$ so that

$$g(t) = \sum_{n=-\infty}^{\infty} D_n x_n(t) \equiv \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad ?$$

$$\Rightarrow D_n = \frac{(g, x_n)}{(x_n, x_n)} = \frac{1}{T_0} \int_{T_0} g(t) e^{-jn\omega_0 t} dt$$

Other forms for Fourier series:

$$C_n \cos[n\omega_0 t + \theta_n] = \underbrace{\left(\frac{C_n}{2} e^{j\theta_n}\right)}_{D_n} e^{jn\omega_0 t} + \underbrace{\left(\frac{C_n}{2} e^{-j\theta_n}\right)}_{D_{-n}} e^{-jn\omega_0 t}$$

$$= \underbrace{[C_n \cos \theta_n]}_{a_n} \cos(n\omega_0 t) + \underbrace{[-C_n \sin \theta_n]}_{b_n} \sin(n\omega_0 t)$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

Fourier spectra for $g(t) = 7 + 40 \cos(2\pi/1000 t + \pi/3)$

"One-sided" spectrum:

"Two-sided" spectrum:

$$\omega_0 =$$

$$\omega_0 =$$

$$C_0 =$$

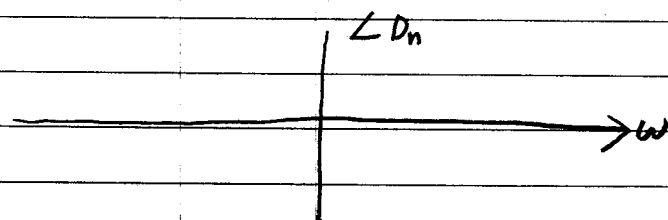
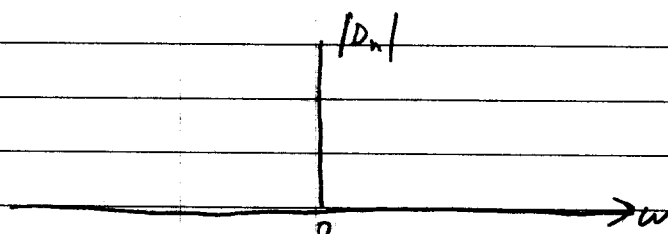
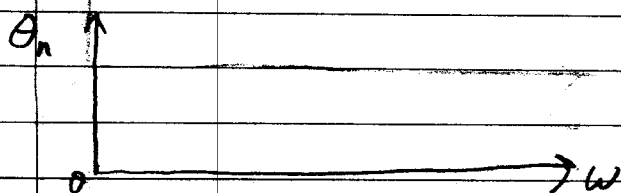
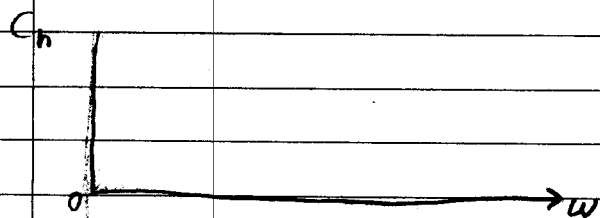
$$D_0 =$$

$$C_1 =$$

$$D_1 =$$

$$\theta_1 =$$

$$D_{-1} =$$



What is "negative frequency"?

$$\cos(2\pi 1000t) = \frac{1}{2} e^{j2\pi 1000t} + \frac{1}{2} e^{-j2\pi 1000t}$$

Real sinusoid has "+" and "-" "frequencies".

⇒ Amplitude is split in half between components.

In modulation, negative frequencies become real & observable!
(We'll see later)

Fourier series computation:

See Examples 2.11 & 2.8.