ELEC 471
Spring, 1999
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Least Squares Modeling of Systems

1 Introduction

The first part of this project is to design a Matlab program to obtain the frequency response of a series RLC circuit and add noise to it to simulate laboratory measurements. Your program should plot the frequency response, the frequency response plus noise (simulated laboratory measurements), and also estimate the resonant frequency, bandwidth, and quality factor of the filter from the noisy measurements. Try various noise distributions and noise power. Your design should allow the user to compare the estimated results with the true values, based on the nominal element values in the circuit. You should also do a least-squares fit between your noisy frequency response and a model for the frequency response.

The series RLC circuit that we will work with is shown below. The resistance r is the internal resistance of the inductor L.

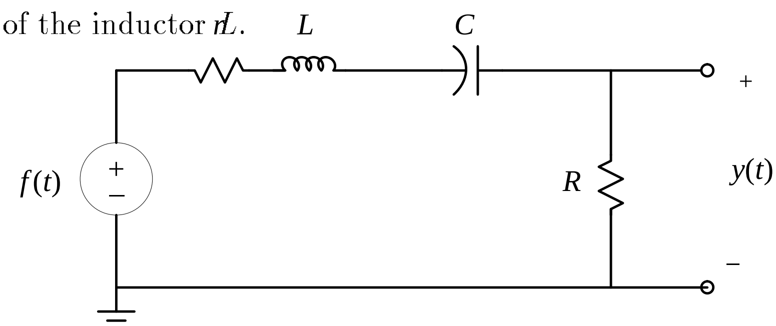

Figure 1: Series RLC circuit.

Please verify that the transfer function of the circuit is

$$H(\omega) = \frac{V_0}{V_1} = \frac{R}{(R+r) + j(\omega L - \frac{1}{\omega C})}.$$

The resonant frequency ω_0, the bandwidth B, and the quality factor Q for this circuit are related to the element values according to

$$\omega_0 = \frac{1}{\sqrt{LC}}, \qquad B = \frac{R+r}{L}, \qquad Q = \frac{\sqrt{L/C}}{R+r}. \tag{1}$$

As a minimum, you should understand how ω_0, B, and Q are related to the shape of the frequency response. In order to really understand these equations, we suggest that you derive them from $H(\omega)$.

2 Least Squares Modeling

Suppose that your circuit is in a box, and all you can do is make input/output measurements. From the bell-shaped appearance of the frequency response, you might hypothesize that the circuit can be modeled by a second-order transfer function of the form

$$|H_{model}(\omega)|^2 = \frac{\omega^2}{X + Y\omega^2 + Z\omega^4}. \tag{2}$$

Equation (2) is a frequency-domain *model* for the system with parameters X, Y, and Z. You should be able to verify that the theoretical transfer function for the RLC circuit has the form in (2), and is

$$|H_{theo}(\omega)|^2 = \frac{\omega^2}{\left(\frac{1}{RC}\right)^2 + \left[\left(\frac{R+r}{R}\right)^2 - \frac{2L}{R^2C}\right]\omega^2 + \left(\frac{L}{R}\right)^2\omega^4}. \tag{3}$$

Our objective in this section is to develop a procedure for choosing the parameters X, Y, and Z in (1) so that the model $|H_{model}(\omega)|^2$ is a good approximation to the noisy frequency response $|H_{meas}(\omega)|^2$ that would be obtained through laboratory measurements. Then, once X, Y, and Z are known, you should be able to to combine (2), (3), and the equations for ω_o, B, and Q in (1) to derive ω_o, B, and Q from X, Y, and Z.

Soving for the optimum X, Y, and Z directly from (2) is difficult, because $|H_{model}(\omega)|^2$ is a *nonlinear* function of X, Y, and Z. Note, however, that the reciprocal is a linear function of the parameters:

$$G_{model}(\omega) \equiv \frac{1}{|H_{model}(\omega)|^2} = X\omega^{-2} + Y + Z\omega^2. \tag{4}$$

Thus it is simpler to find X, Y, and Z that make $G_{model}(\omega)$ a "best approximation" to the reciprocal of the measured data, $G_{meas}(\omega) \equiv 1/\left|H_{meas}(\omega)\right|^2$.

By "best approximation" we mean that the total squared error between the experimental data $G_{meas}(\omega)$ and the model $G_{model}(\omega)$ is as small as possible. The measured frequency response is only available at N frequency values, and let us denote those (radian) frequency values by $\omega_1, \omega_2, \ldots, \omega_N$. Then total squared error is defined as follows:

$$\begin{aligned}
e_n &= G_{model}(\omega_n) - G_{meas}(\omega_n) = \text{error at frequency } \omega_n && (5)\\
e_n^2 &= [G_{model}(\omega_n) - G_{meas}(\omega_n)]^2 = \text{squared error at frequency } \omega_n && (6)\\
E &= \sum_{n=1}^{N} e_n^2 = \text{total squared error} \\
&= \sum_{n=1}^{N} [G_{model}(\omega_n) - G_{meas}(\omega_n)]^2 \\
&= \sum_{n=1}^{N} \left[X\omega_n^{-2} + Y + Z\omega_n^2 - G_{meas}(\omega_n)\right]^2 && (7)
\end{aligned}$$

The parameters X, Y, and Z are chosen to make E in (7) as small as possible.

Solving for the best X, Y, and Z is a multivariable optimization problem. However, it is the simplest type of such problems, since E in (7) is a three-dimensional parabola with a single, global minimum. We must find the X, Y, and Z that correspond to the minimum of the parabola in (7). This is easily done by setting the partial derivatives equal to zero:

$$\frac{\partial E}{\partial X} = \sum_{n=1}^{N} 2\omega_n^{-2}\left[X\omega_n^{-2} + Y + Z\omega_n^2 - G_{meas}(\omega_n)\right] = 0 \qquad (8)$$

$$\frac{\partial E}{\partial Y} = \sum_{n=1}^{N} 2\left[X\omega_n^{-2} + Y + Z\omega_n^2 - G_{meas}(\omega_n)\right] = 0 \qquad (9)$$

$$\frac{\partial E}{\partial Z} = \sum_{n=1}^{N} 2\omega_n^{2}\left[X\omega_n^{-2} + Y + Z\omega_n^2 - G_{meas}(\omega_n)\right] = 0 \qquad (10)$$

Now (8), (9), and (10) are a set of three linear equations in the three unknowns X, Y, and Z. One way to solve them is Gaussian elimination. Matlab will solve the equations very easily if we first formulate (8), (9), and (10) as the following equivalent matrix equation:

$$\begin{bmatrix} \sum_{n=1}^N \omega_n^{-4} & \sum_{n=1}^N \omega_n^{-2} & N \\ \sum_{n=1}^N \omega_n^{-2} & N & \sum_{n=1}^N \omega_n^{2} \\ N & \sum_{n=1}^N \omega_n^{2} & \sum_{n=1}^N \omega_n^{4} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^N \omega_n^{-2} G_{meas}(\omega_n) \\ \sum_{n=1}^N G_{meas}(\omega_n) \\ \sum_{n=1}^N \omega_n^{2} G_{meas}(\omega_n) \end{bmatrix} \qquad (11)$$

Equation (11) has the general form

$$\mathbf{M}\mathbf{x} = \mathbf{b}, \qquad (12)$$

where $\mathbf{M}$ is a 3×3 matrix computed from the measured data, $\mathbf{x}$ is the 3×1 vector of unknowns X, Y, Z, and $\mathbf{b}$ is the 3×1 vector on the right side of (11), which is also computed from the measured data. Once $\mathbf{M}$ and $\mathbf{b}$ are defined in Matlab, you can solve for $\mathbf{x}$ by Gaussian elimination with the simple Matlab command

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>> x = M \ b
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Then you can plot the least-squares fit $|H_{model}(\omega)|^2 = 1/G_{model}(\omega)$ using the values of X, Y, Z obtained by solving (11).

A practical problem arises when you try to compute the least-squares solution according to (11). Note that the formulation involves quantities of the form ω_n^4 as well as quantities of the form ω_n^{-4}. For $\omega = 10^4$, which is quite reasonable in your measured data, this leads to magnitudes on the order of $\omega^4 = 10^{16}$ and $\omega^{-4} = 10^{-16}$. Computers find it hard to manipulate very large numbers and very small numbers in the same problem.

A way out of this numerical difficulty is to *normalize* your frequency values so they span a smaller range. Choose a normalizing frequency k, and define $v = \omega/k$. Then substitute $\omega = vk$ in (2) and define new variables X', Y', and Z' so that (2) can be written in the form

$$|H_{model}(v)|^2 = \frac{v^2}{X' + Y'v^2 + Z'v^4}. \qquad (13)$$

Then use exactly the same solution as given above, with ω replaced by v and X, Y, Z replaced by X', Y', and Z'. Once X', Y', and Z' are computed, relate them back to X, Y, Z. Then the desired quantities ω_0, B, and Q can be calculated from X, Y, Z.

3 Project Report Guidelines

Details will be provided later about specific tasks for this project.