ELEC 471
Prof. Rich Kozick

# Homework 9

Date Assigned: Thursday, April 5, 2001
Date Due: Thursday, April 12, 2001

**Reading:** Please read Chapter 4, Sections 4.1 - 4.5.

1. In this problem, I would like you to perform an analysis of a digital communication system that is similar to the case that we studied in class on April 5. The only difference is that the signal $S$ and noise $N$ have PMFs as follows.

$$P_S(s) = \begin{cases} 0.25, & s = -1 \\ 0.75, & s = +1 \\ 0, & \text{otherwise} \end{cases} \qquad P_N(n) = \begin{cases} 0.5, & n = -1 \\ 0.25, & n = 0 \\ 0.25, & n = +1 \\ 0, & \text{otherwise} \end{cases}$$

   The random variables $S$ and $N$ are independent. The receiver observes the random variable $X = S + N$.

   (a) Suppose the receiver obtains the value $X = x$. Find the signal estimate $\hat{s}(x)$ that minimizes the mean-squared error $E[(S - \hat{s}(x))^2]$. Show all of the steps in your analysis, and display $\hat{s}(x)$ as a plot versus $x$. I suggest that you draw plots of $P_S(s), P_N(n), P_{X,S}(x, s), P_X(x), P_{S|X}(s|x)$ as we did in class.

   (b) How would you process the signal estimates $\hat{s}(x)$ in order to recover the *binary* values $\{-1, +1\}$ of $S$? In other words, what "decision rule" would you use to recover the bits from $X$ based on $\hat{s}(x)$?

   (c) For the decision rule that you developed in part b, what is the probability of a bit error for this system? Explain your reasoning.

   (d) Write a MATLAB program to simulate this system, including the decision rule developed in part b. Compare the bit error rate (BER) in your simulation with the analytical probability of a bit error computed in part c.

   **Be sure to compare the simulated and analytical BER!**

2. Please answer the following questions for the joint PMF $P_{X,Y}(x, y)$ shown in the figure below.

   (a) Find the marginal PMFs $P_X(x)$ and $P_Y(y)$, and plot them.

   (b) Find the mean and variance of $X$ and $Y$: $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$.

   (c) Find the correlation between $X$ and $Y$, defined as $r_{X,Y} = E[XY]$.

   (d) Find the covariance $\text{Cov}[X, Y]$.

(e) Find the correlation coefficient $\rho_{X,Y}$.

(f) Are $X$ and $Y$ *independent* random variables? Recall that $X$ and $Y$ are independent if and only if $P_{X,Y}(x,y) = P_X(x)P_Y(y)$.

(g) Find the conditional PMFs

$$P_{X|Y}(x|y) = P[X = x|Y = y]$$

$$P_{Y|X}(y|x) = P[Y = y|X = x]$$

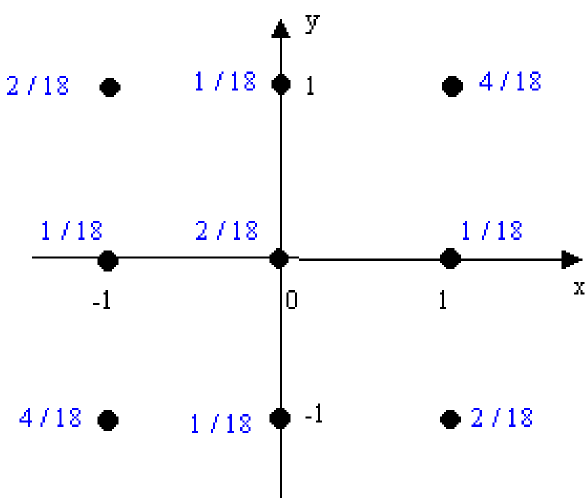
and plot each of these versus $x$ and $y$ (on separate plots).

(h) Suppose you need to produce an estimate $\hat{x}$ of $X$ that minimizes the mean squared error $E[(X - \hat{x})^2]$. You must do this with *no information* about $Y$. What value should you choose for $\hat{x}$, and why? (Your answer should be a number!)

(i) Suppose that you *observe* that the random variable $Y$ takes on the value $y$ (i.e., $Y = y$). We would like to incorporate the knowledge that $Y = y$ to improve our estimate of $X$. For each possible value of $y$, what is your estimate of $X$, denoted by $\hat{x}(y)$? Explain how to compute $\hat{x}(y)$, and present a plot of $\hat{x}(y)$ versus $y$.

(j) Is $\hat{x}(y)$ in part (i) different from $\hat{x}$ in part (h)? Is this reasonable based on the probability values in the joint PMF? Is this reasonable based on the value of correlation coefficient $\rho_{X,Y}$ that you computed in part (e)? That is, does the value of $\rho_{X,Y}$ lead you to expect that information about $Y$ should be useful in predicting the value of $X$?

(k) Using your answer from part (h), compute $E[(X - \hat{x})^2]$.
Using your answer from part (i), compute
$E[(X - \hat{x}(-1))^2|Y = -1]$, $E[(X - \hat{x}(0))^2|Y = 0]$, and $E[(X - \hat{x}(1))^2|Y = 1]$. Do these results show that we get better estimates for $X$ when the value of $Y$ is known? Please explain.


y
2 / 18
1 / 18
1
4 / 18
1 / 18
2 / 18
1 / 18
-1
0
1
x
4 / 18
1 / 18
-1
2 / 18