Probability and Stochastic Processes

A Friendly Introduction for Electrical and Computer Engineers

Chapter 3 Viewgraphs

Multiple (2) random variables

- An experiment produces both X and Y
- Example:
  - $X =$ minutes you wait a Rutgers bus
  - $Y =$ no. of other Rutgers buses pass by
- Joint range:

$$S_{X,Y} = \{(x,y) | P_{X,Y}(x,y) > 0\} \subset S_X \times S_Y$$

Probability Model for 2 RV's

- The *joint probability mass function* of X and Y is

$$P_{X,Y}(x, y) = P[X = x, Y = y]$$

- Joint PMF is a rule that for any x and y, gives the probability that $X = x$ and $Y = y$.

Example

- Flip a coin three times.
- $X =$ no. of H
- $Y =$ no. of T after the first H
- Write all 8 outcomes: $\{HHH, \ldots, TTT\}$.
- For each outcome, record X and Y

Marginal PMF

- Experiment produces X and Y.
- All you care about is X.
- PMF of X is

$$P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x, y)$$

- Calculate by summing across rows, write PMF in margin

Functions of 2 RVs

- Derived rv $W = g(X, Y)$
- PMF of W:

$$P_W(w) = \sum_{(x,y):g(x,y)=w} P_{X,Y}(x, y)$$

Derived PMF Example

Example $W = XY$:

Expectations of Functions again

- Thm: The expected value of $W = g(X, Y)$ is

$$E[W] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y) P_{X,Y}(x, y)$$

- If $g(X, Y) = g_1(X, Y) + g_2(X, Y) + \cdots + g_k(X, Y)$, then

$$E[g(X, Y)] = E[g_1(X, Y)] + \cdots + E[g_n(X, Y)]$$

Expectations of Sums

- $E[X + Y] = E[X] + E[Y]$
- With $W = (X + Y - \mu_{X+Y})^2$,

$$\begin{aligned} E[W] &= \mathrm{Var}\,[X + Y] \\ &= \mathrm{Var}\,[X] + \mathrm{Var}\,[Y] + 2E[(X - \mu_X)(Y - \mu_Y)] \end{aligned}$$

Covariance

- Covariance: $W = (X - \mu_X)(Y - \mu_Y)$,

$$\mathrm{Cov}\,[X, Y] = E[W] = E[(X - \mu_X)\,(Y - \mu_Y)]$$

- Covariance is also

$$\mathrm{Cov}\,[X, Y] = E[XY] - \mu_X \mu_Y$$

- $\mathrm{Cov}\,[X, Y] > 0$ says, $X > E[X]$ implies $Y > E[Y]$ is likely (X goes up, Y goes up)

- $\mathrm{Var}\,[X + Y] = \mathrm{Var}\,[X] + \mathrm{Var}\,[Y] + 2\,\mathrm{Cov}\,[X, Y]$

Correlation

- The *correlation* of X and Y is $E[XY]$
- Correlation = Covariance if $E[X] = E[Y] = 0$
- $E[XY] > 0$ suggests that $X > 0$ increases chance $Y > 0$
- **Orthogonal**: $E[XY] = 0$
- **Uncorrelated**: $\mathrm{Cov}\,[X, Y] = 0$

Correlation Coefficient

- The *correlation coefficient* of two random variables X and Y is

$$\rho_{X,Y} = \frac{\mathrm{Cov}\left[X, Y\right]}{\sqrt{\mathrm{Var}\left[X\right]\mathrm{Var}\left[Y\right]}}$$

- Thm: the Correlation coefficient is normalized:

$$-1 \leq \rho_{X,Y} \leq 1$$

- Thm: If $Y = aX + b$, then

$$\rho_{X,Y} = \begin{cases} -1 & a < 0 \\ 0 & a = 0 \\ 1 & a > 0 \end{cases}$$

Conditioning Events (Again!)

- The *conditional joint PMF* of X and Y given B is

$$P_{X,Y|B}(x,y) = P[X = x, Y = y|B]$$

- If B is an event on the X, Y plane,

$$P[(X = x, Y = y) \cap B] = \begin{cases} P[X = x, Y = y] & (x,y) \in B \\ 0 & \text{otherwise} \end{cases}$$

Conditional Expectations

- The *conditional expected value of* $g(X,Y)$ *given* B is

$$E[g(X,Y)|B] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x,y) P_{X,Y|B}(x,y)$$

Conditional PMF of X given Y

- For any event $Y = y$ such that $P_Y(y) > 0$, the *conditional PMF of* X *given* $Y = y$ is

$$P_{X|Y}(x|y) = P[X = x|Y = y]$$

- If $P_X(x) > 0$ and $P_Y(y) > 0$,

$$P_{X,Y}(x, y) = P_{X|Y}(x|y)\, P_Y(y) = P_{Y|X}(y|x)\, P_X(x)$$

Conditional Expected Values

- For any $y \in S_Y$, the *conditional expected value* of $g(X)$ given $Y = y$ is

$$E[g(X)|Y = y] = \sum_{x \in S_X} g(x) P_{X|Y}(x|y)$$

- Special cases
  - $E[X|Y = y]$
  - Cond. Variance:

$$\text{Var}[X|Y = y] = E\left[(X - E[X|Y = y])^2 \,|Y = y\right]$$

Conditional Expectation

- The conditional expectation function $E[X|Y]$ is a function of random variable Y such that if $Y = y$ then $E[X|Y] = E[X|Y = y]$
- Let $g(y) = E[X|Y = y]$. When $Y = y$, $E[X|Y]$ takes on the value $g(y)$. That is, $E[X|Y] = g(Y)$

Independent random variables

- X and Y are independent if and only if $\{X = x\}$ and $\{Y = y\}$ are independent events for all x and y

- Equivalently, X and Y are independent if

$$P_{X,Y}(x, y) = P_X(x)\, P_Y(y)$$

Properties of Independent RVs

- For independent random variables X and Y,
  - $r_{X,Y} = E[XY] = E[X]E[Y]$
  - $E[X|Y = y] = E[X]$ for all $y \in S_Y$
  - $E[Y|X = x] = E[Y]$ for all $x \in S_X$
  - $\text{Var}\,[X + Y] = \text{Var}\,[X] + \text{Var}\,[Y]$
  - $\text{Cov}\,[X, Y] = \rho_{X,Y} = 0$

Multiple Discrete RVs

- The joint PMF of the discrete random variables $X_1, \ldots, X_n$ is

$$P_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = P[X_1 = x_1, \ldots, X_n = x_n]$$