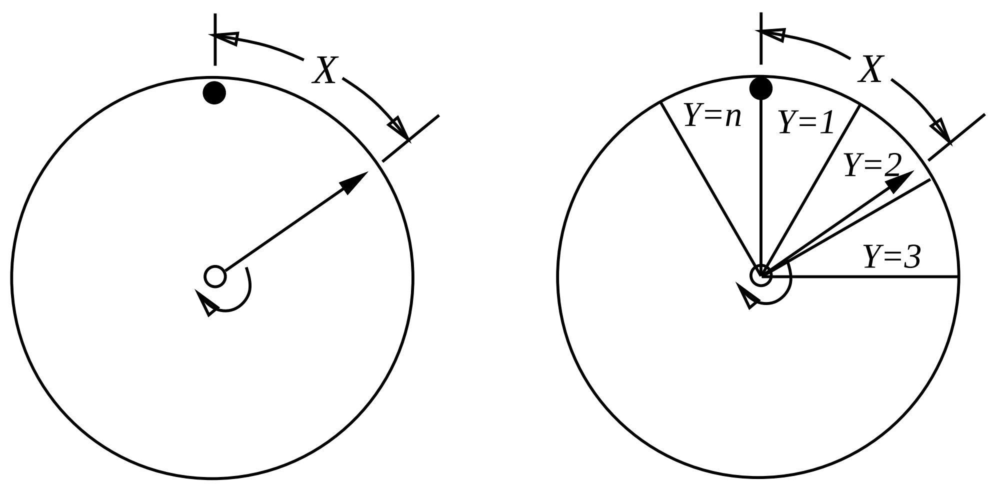
Probability and Stochastic Processes

A Friendly Introduction for Electrical and Computer Engineers

Chapter 4 Viewgraphs

Continuous Random Variables

- Example: Spin wheel with circumference $c = 1$. Measure distance around circumference



Discretized Wheel

- With n slices,

$$P_Y(y) = \begin{cases} 1/n & y = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases} .$$

- if $X = x$, then $Y = \lceil nx \rceil$ or $\{X = x\} \subset \{Y = \lceil nx \rceil\}$, so that

$$P[X = x] \leq P[Y = \lceil nx \rceil] = \frac{1}{n}.$$

- As n increases,

$$P[X = x] \leq \lim_{n \to \infty} P[Y = \lceil nx \rceil] = \lim_{n \to \infty} \frac{1}{n} = 0.$$

- Thus for any x,

$$P[X = x] = 0$$

X is **not** a discrete rv!

CDF for continuous RVs

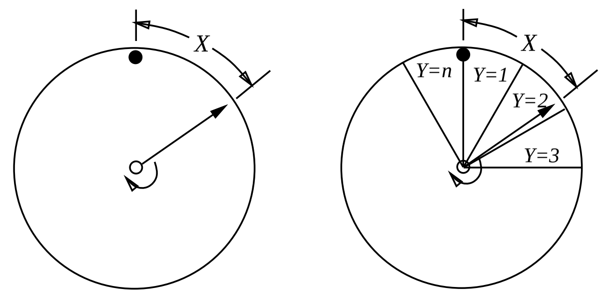
- CDF Definition:

$$F_X(x) = P[X \leq x]$$

- Definition: X is a *continuous random variable* if the CDF $F_X(x)$ is a continuous function.

CDF of the Wheel Pointer

- For the Wheel pointer:



$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

Probability Density Function (PDF)

- To see relative likelihoods of possible values:

$$P[x_1 < X \leq x_1 + \Delta] = \frac{F_X(x_1 + \Delta) - F_X(x_1)}{\Delta} \Delta$$

- As $\Delta \to 0$,

$$\lim_{\Delta \to 0} \frac{F_X(x_1 + \Delta) - F_X(x_1)}{\Delta} = f_X(x_1)$$

- Definition: PDF $f_X(x) = dF_X(x)/dx$ and

$$P[x_1 < X \leq x_1 + \Delta] \approx f_X(x_1)\,\Delta$$

PDF Properties

- $f_X(x) \geq 0$ for all x
- $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$
- $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
- $P[x_1 < X \leq x_2] = \int_{x_1}^{x_2} f_X(x)\, dx$

Expected Values

- If Y is discrete, $E[Y] = \sum_{y_i \in S_Y} y_i P_Y(y_i)$
- Discretize continuous rv X: $Y = \Delta \left\lfloor \frac{X}{\Delta} \right\rfloor$
- $Y = k\Delta$ iff $k\Delta \leq X < k\Delta + \Delta$

$$\begin{aligned} E[Y] &= \sum_{k=-\infty}^{\infty} k\Delta P[Y = k\Delta] \\ &= \sum_{k=-\infty}^{\infty} k\Delta P[k\Delta \leq X < k\Delta + \Delta] \end{aligned}$$

Expected Values 2

- As $\Delta \to 0$, $Y \to X$ and

$$P[k\Delta \leq X < k\Delta + \Delta] \to f_X(k\Delta)\,\Delta$$

- so that for small Δ,

$$E[X] \approx \sum_{k=-\infty}^{\infty} k\Delta f_X(k\Delta)\,\Delta \to \int_{-\infty}^{\infty} x f_X(x)\; dx$$

Expected Value of a Function

- The expected value of $g(X)$ is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

Properties of Expected Values

- $E[X - \mu_X] = 0$
- $E[aX + b] = aE[X] + b$
- $\mathrm{Var}\,[X] = E\left[X^2\right] - (E[X])^2$
- If $X = a$ always, then $\mathrm{Var}\,[X] = 0$
- The variance of $Y = aX + b$ is $\mathrm{Var}\,[Y] = a^2\,\mathrm{Var}\,[X]$

Uniform Random Variable

$$f_X(x) = \begin{cases} 1/(b-a) & a \leq x < b \\ 0 & \text{otherwise} \end{cases}$$

where the two parameters are $b > a$.

Exponential Random Variable

$$f_X(x) = \begin{cases} ae^{-ax} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where the parameter $a > 0$.

Erlang Random Variable

$$f_X(x) = \begin{cases} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where the parameter $\lambda > 0$, and the parameter $n \geq 1$ is an integer.

Gaussian RVs

- X is a Gaussian random variable if

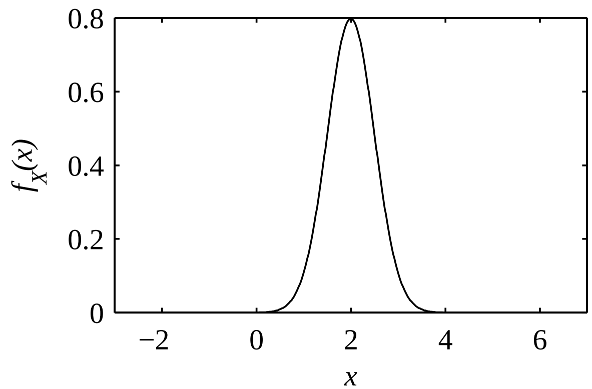
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

  where μ is real and $\sigma > 0$.

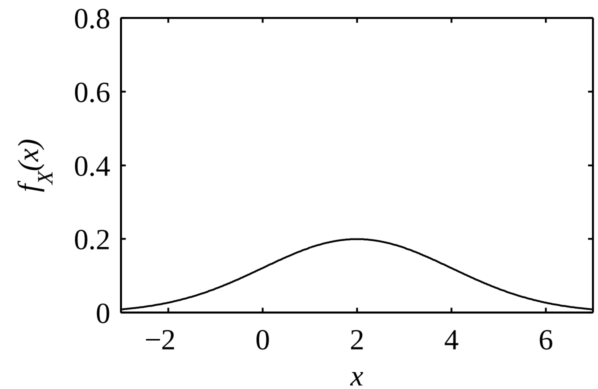
- Notation: X is $N[\mu, \sigma^2]$

- Properties: $E[X] = \mu$, $\text{Var}[X] = \sigma^2$

2 Gaussian PDFs



(a) $\mu = 2, \sigma = 1/2$



(b) $\mu = 2, \sigma = 2$

Standard Normal Random Variable

- Z is a *standard normal random variable* if Z is $N[0,1]$
- The CDF of Z is

$$\Phi(z) = \int_{-\infty}^{z} f_Z(u)\, du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du$$

- Tables for $\Phi(z)$ and $Q(z) = 1 - \Phi(z)$

Linear Transformation

- X is $N[\mu, \sigma^2]$, $\implies Y = aX + b$ is $N[a\mu + b, \sigma\sqrt{a}]$
- Thus $Z = \dfrac{X - \mu_X}{\sigma_X}$ is $N[0, 1]$ and

$$P[X \leq x] = P\left[\frac{X - \mu_X}{\sigma_X} \leq \frac{x - \mu_X}{\sigma_X}\right]$$

or

$$F_X(x) = P\left[Z \leq \frac{x - \mu_X}{\sigma_X}\right] = \Phi\left(\frac{x - \mu_X}{\sigma_X}\right)$$

PDFs for Discrete Random Variables

- X discrete: $P_X(x_i)$
- In terms of unit step $u(x)$, CDF is

$$F_X(x) = \sum_i P_X(x_i)\, u(x - x_i)$$

- PDF of X is

$$f_X(x) = \frac{dF_X(x)}{dx} = \sum_i P_X(x_i)\, \delta(x - x_i)$$

Mixed Random Variables

- CDF $F_X(x)$ is piecewise cts but has jumps at $x_1, x_2, \ldots$
- Jump at x_i is $P[X = x_i]$
- PDF has impulses at x_i weighted by $P[X = x_i]$

Problem 4.6.6

- phone call: $P[\text{line busy}] = 0.2$, $P[\text{no answer}] = 0.3$
- $X =$ duration of a completed phone call, exponential with $E[X] = 3$
- $W =$ duration of any call ($W = 0$ if line busy or no answer)
- Find $F_W(w)$ and $f_W(w)$?

Derived Probability Models

- PDF of $Y = g(X)$?
- Two step procedure:

1. Find the CDF $F_Y(y) = P[g(X) \leq y]$
2. Take the derivative to get the PDF

- This can be tricky!

Problem 4.7.5

U is a uniform over $[0, 1]$ and $X = -\ln(1 - U)$.

- Find $f_X(x)$

Conditional PDF given an Event

- Given X with PDF $f_X(x)$ and event $B \subset S_X$ with $P[B] > 0$,
- *Conditional PDF of X given B* is

$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases}$$

- Conditional Expectations: (Use $f_{X|B}(x)$)

$$E[X|B] \quad E[g(X)|B] \quad \mathrm{Var}[X|B]$$