Probability and Stochastic Processes

A Friendly Introduction for Electrical and Computer Engineers

Chapter 1 Viewgraphs

Set Theory Preliminaries

- Venn Diagrams
- Universal Set/Empty Set
- Union/Intersection
- Complement
- Mutually Exclusive/Collectively Exhaustive

What is Probability

- a number between 0 and 1.
- a physical property (like mass or volume) that can be measured?
- Measure of our knowledge?

Experiments

- Procedure + Observations
- Real Experiments are TOO complicated
- Instead we analyze/develop models of experiments
  - A coin flip is equally likely to be H or T

Example 1.1

An experiment consists of the following procedure, observation and model:

- Procedure: Flip a coin and let it land on a table.
- Observation: Observe which side (head or tail) faces you after the coin lands.
- Model: Heads and tails are equally likely. The result of each flip is unrelated to the results of previous flips.

Definition Outcome, Sample Space

An *outcome* of an experiment is any possible observation of that experiment.

The *sample space* of an experiment is the finest-grain, mutually exclusive, collectively exhaustive set of all possible outcomes. An *event* is a set of outcomes of an experiment.

Correspondences

Set Algebra	Probability
set	event
universal set	sample space
element	outcome

Example 1.9

Flip four coins, a penny, a nickel, a dime, and a quarter. Examine the coins in order (penny, then nickel, then dime, then quarter) and observe whether each coin shows a head (h) or a tail (t). What is the sample space? How many elements are in the sample space?

. . . .

The sample space consists of 16 four-letter words:

$$\{tttt, ttth, ttht, \ldots, hhhh\}$$

Event Spaces

- An *event space* is a collectively exhaustive, mutually exclusive set of events.

- **Example 1.10**: For $i = 0, 1, 2, 3, 4$,

$$B_i = \{\text{outcomes with } i \text{ heads}\}$$

- Each B_i is an event containing one or more outcomes:

- The set $B = \{B_0, B_1, B_2, B_3, B_4\}$ is an event space.

Theorem 1.2

- For an event space $B = \{B_1, B_2, \ldots\}$ and any event A, let $C_i = A \cap B_i$.
- For $i \neq j$, the events $C_i \cap C_j = \phi$.
- $$A = C_1 \cup C_2 \cup \cdots$$

Axioms of Probability

A probability measure $P[\cdot]$ is a function that maps events in the sample space to real numbers such that

Axiom 1 For any event A, $P[A] \geq 0$.

Axiom 2 $P[S] = 1$.

Axiom 3 For any countable collection $A_1, A_2, \ldots$ of mutually exclusive events

$$P[A_1 \cup A_2 \cup \cdots] = P[A_1] + P[A_2] + \cdots$$

Consequences of the Axioms

Theorem 1.4: If

$$B = B_1 \cup B_2 \cup \cdots \cup B_m$$

and for $i \neq j$,

$$B_i \cap B_j = \phi$$

then

$$P[B] = \sum_{i=1}^{m} P[B_i]$$

Problem 1.3.5

A student's score on a 10-point quiz is equally likely to be any integer between 0 and 10. What is the probability of an A, which requires the student to get a score of 9 or more? What is the probability the student gets an F by getting less than 4?

Consequences of the Axioms

Theorem 1.7: The probability measure $P[\cdot]$ satisfies

- $P[\phi] = 0$.
- $P[A^c] = 1 - P[A]$.
- For any A and B,

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

- If $A \subset B$, then $P[A] \leq P[B]$.

Problem 1.4.5

A cellphone is equally likely to make zero handoffs (H_0), one handoff (H_1), or more than one handoff (H_2). Also, a caller is on foot (F) with probability $5/12$ or in a vehicle (V).

- Find three ways to fill in the following probability table:

	H_0	H_1	H_2
F			
V			

- If $1/4$ of all callers are on foot making calls with no handoffs and that $1/6$ of all callers are vehicle users making calls with a single handoff, what is the table?

Conditioning

- $P[A] =$ our knowledge of the likelihood of A
- $P[A] =$ a priori probability
- Suppose we cannot completely observe an experiment
  - We learn that event B occurred
  - We do not learn the precise outcome

Conditional Probability Definition

- Learning B occurred changes $P[A]$
- The *conditional probability* A given the occurrence of B is

$$P[A|B] = \frac{P[AB]}{P[B]}$$

Problem 1.5.6

For deer ticks in the Midwest,

- 16% carried Lyme disease (event L)
- 10% had HGE (event H)
- 10% of the ticks that had either Lyme or HGE carried both

Find $P[LH]$ and then $P[H|L]$.

Law of Total Probability

- If $B_1, B_2, \ldots, B_m$ is an event space and $P[B_i] > 0$ for $i = 1, \ldots, m$, then
$$P[A] = \sum_{i=1}^{m} P[A|B_i]P[B_i]$$

Bayes' Theorem

- $$P[B|A] = \frac{P[A|B]P[B]}{P[A]}$$

- For an event space $B_1, B_2, \ldots, B_m$,

$$P[B_i|A] = \frac{P[A|B_i]P[B_i]}{\sum_{i=1}^{m} P[A|B_i]P[B_i]}$$

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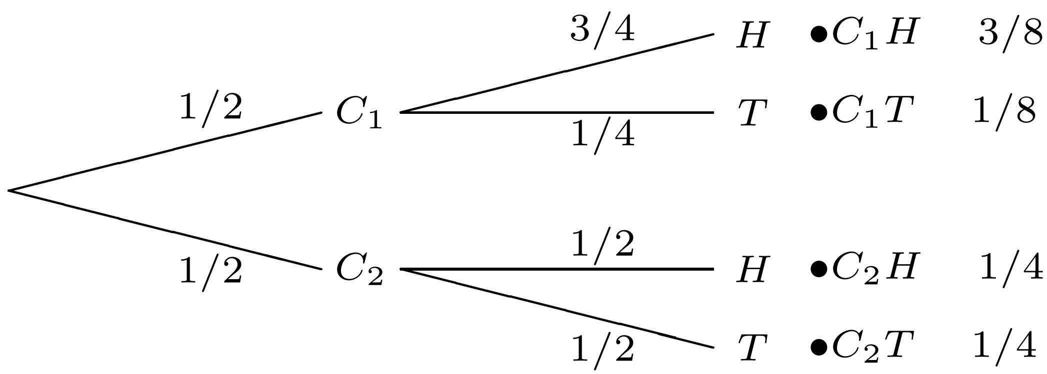
Sequential Experiments - Example

- Two coins, one biased, one fair, but you don't know which is which.
- Coin 1: $P[H] = 3/4$. Coin 2: $P[H] = 1/2$
- Pick a coin at random and flip it. Let C_i denote the event that coin i is picked. What is $P[C_1|H]$

Solution: Tree Diagram



$$P[C_1|H] = \frac{P[C_1H]}{P[C_1H] + P[C_2H]} = \frac{3/8}{3/8 + 1/4} = \frac{3}{5}$$

Definition 2 Independent Events

Definition 1.6 Events A and B are *independent* if and only if

$$P[AB] = P[A]P[B] \tag{1}$$

. . . .

Equivalent definitions:

$$P[A|B] = P[A] \qquad P[B|A] = P[B]$$

Always check if you are asked!

Definition 3 Independent Events

Definition 1.7 A_1, A_2, and A_3 are *independent* if and only if

- A_1 and A_2 are independent.
- A_2 and A_3 are independent.
- A_1 and A_3 are independent.
- $P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$.

Fundamental Principle of Counting

- Experiment A has n possible outcomes,
- Experiment B has k possible outcomes,
- There are nk possible outcomes when you perform both experiments.

Permutations

- k-*permutation*: an ordered sequence of k distinguishable objects
- $(n)_k$ = no. of k-permutations of n dist. objects.

$$(n)_k = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Combinations

- Pick a subset of k out of n objects.
- Order of selection doesn't matter
- Each subset is a k-*combination*

How Many Combinations

- $\binom{n}{k}$ = "n choose k"
- Two steps for a k-permutation:

1. Choose a k-combination out of n objects.
2. Choose a k-permutation of the k objects in the k-combination.

$$(n)_k = \binom{n}{k} k! \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Problem 1.8.6

A basketball team has

- 3 pure centers, 4 pure forwards, 4 pure guards
- one swingman who can play either guard or forward.

A pure player can play only the designated position. How many lineups are there (1 center, 2 forwards, 2 guards)

Problem 1.8.6 Solution

Three possibilities:

1. swingman plays guard: N_1 lineups
2. swingman plays forward N_2 lineups
3. swingman doesn't play. N_3 lineups

$$N = N_1 + N_2 + N_3$$

$$N_1 = \binom{3}{1}\binom{4}{2}\binom{4}{1} = 72$$

$$N_2 = \binom{3}{1}\binom{4}{1}\binom{4}{2} = 72$$

$$N_3 = \binom{3}{1}\binom{4}{2}\binom{4}{2} = 108$$

Multiple Outcomes

- n independent trials
- r possible trial outcomes $(s_1, \ldots, s_r)$
- $P[s_k] = p_k$

Multiple Outcomes (2)

- Outcome is a sequence:
  - Example: $s_3 s_4 s_3 s_1$

$$
\begin{aligned}
P[s_3 s_4 s_3 s_1] = p_3 p_4 p_3 p_1 &= p_1 p_3^2 p_4 \\
&= p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}
\end{aligned}
$$

- Prob depends on how many times each outcome occurred

Multiple Outcomes (3)

N_i = no. of time s_i occurs

$$P[N_1 = n_1, \ldots, N_r = n_r] = M p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

$$\begin{aligned} M &= \text{Multinomial Coefficient} \\ &= \frac{n!}{n_1! n_2! \cdots n_r!} \end{aligned}$$