Probability and Stochastic Processes

A Friendly Introduction for Electrical and Computer Engineers

Chapter 2 Viewgraphs

Random Variables

- Experiment: Procedure + Observations
- Observation is an outcome
- Assign a number to each outcome: Random variable

Random Variables

Three ways to get a rv:

- The rv is the observation
- The rv is a function of the observation
- The rv is a function of a rv

Discrete Random Variables

- $S_X =$ range of X (set of possible values)
- X is discrete is S_X is countable
- Discrete rv X has PMF

$$P_X(x) = P[X = x]$$

PMF Properties

- $P_X(x) \geq 0$
- $\sum_{x \in S_X} P_X(x) = 1$
- For an event $B \subset S_X$,

$$P[B] = P[X \in B] = \sum_{x \in B} P_X(x)$$

Bernoulli RV

Get the phone number of a random student. Let $X = 0$ if the last digit is even. Otherwise, let $X = 1$.

$$P_X(x) = \begin{cases} 1 - p & x = 0 \\ p & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

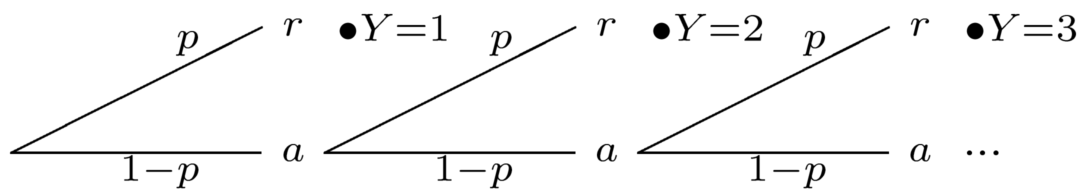
Binomial RV

- Test n circuits, each circuit is rejected with probability p independent of other tests.
- $K =$ no. of rejects
- K is the number of successes in n trials:

$$P_K(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

Geometric RV

Circuit rejected with prob p. Y is the number of tests up to and including the first reject.

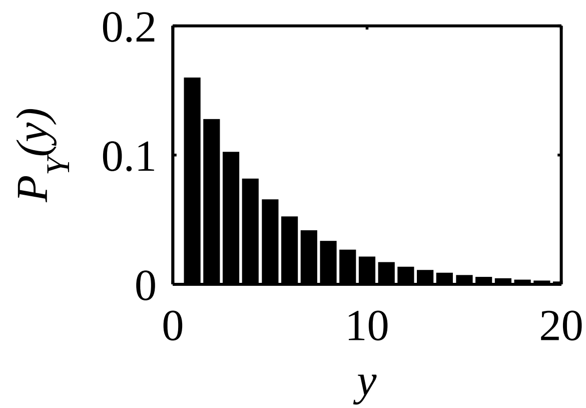


From the tree, $P[Y = 1] = p$, $P[Y = 2] = p(1 - p)$,

$$P_Y(y) = \begin{cases} p(1-p)^{y-1} & y = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Geometric: $p = 0.2$

$$P_Y(y) = \begin{cases} (0.2)(0.8)^{y-1} & y = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$



Pascal RV

- No. of tests, L, needed to find k rejects.

$$P[L = l] = P[AB]$$

- $A = \{k - 1 \text{ rejects in } l - 1 \text{ tests}\}$
- $B = \{\text{success on attempt } l\}$
- Events A and B are independent

Pascal continued

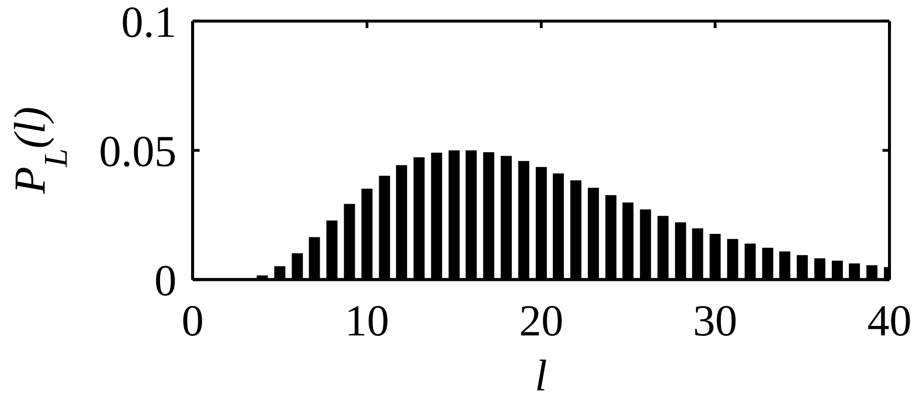
- $P[B] = p$ and $P[A]$ is binomial:

$$\begin{aligned} P[A] &= P[k-1 \text{ succ. in } l-1 \text{ trials}] \\ &= \binom{l-1}{k-1} p^{k-1} (1-p)^{l-1-(k-1)} \end{aligned}$$

$$\begin{aligned} P_L(l) &= P[A]P[B] \\ &= \begin{cases} \binom{l-1}{k-1} p^k (1-p)^{l-k} & l = k, k+1, \ldots \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Pascal: $p = 0.2$, $k = 4$

$$P_L(l) = \begin{cases} \binom{l-1}{3}(0.2)^4(0.8)^{l-4} & l = 4, 5, \ldots \\ 0 & \text{otherwise.} \end{cases}$$



Summary

- **Bernoulli** No. of succ. on one trial
- **Binomial** No. of succ on n trials
- **Geometric** No. of trials until first succ.
- **Pascal** No. of trials until succ k

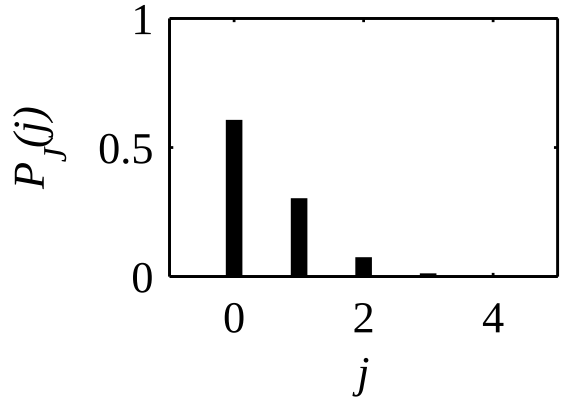
Poisson rv

- Counts *arrivals* of something.
- Arrival rate λ, interval time T.
- With $\alpha = \lambda T$,

$$P_X(x) = \begin{cases} \alpha^x e^{-\alpha}/x! & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

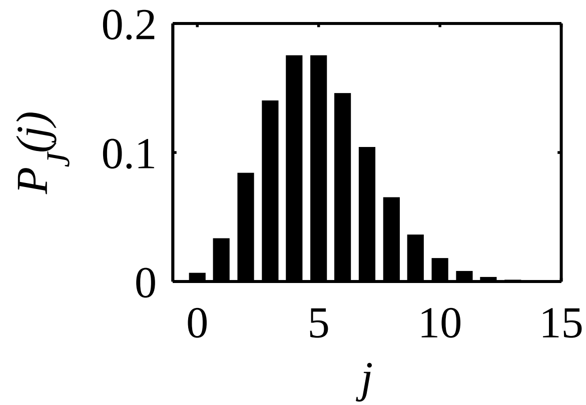
Poisson: $\alpha = 0.5$

$$P_J(j) = \begin{cases} (0.5)^j e^{-0.5}/j! & j = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$



Poisson: $\alpha = 5$

$$P_J(j) = \begin{cases} 5^j e^{-5}/j! & j = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

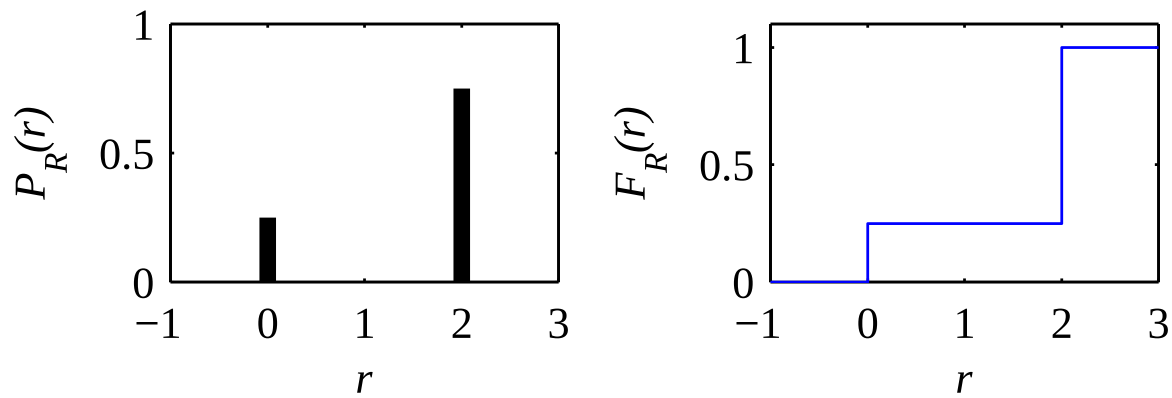


Cumulative Distribution Functions

- The *cumulative distribution function* (CDF) of random variable X is

$$F_X(x) = P[X \leq x]$$

CDF Example



At the discontinuities $r = 0$ and $r = 2$, $F_R(r)$ is the upper values. (right hand limit)

CDF Properties

For any discrete rv X, range $S_X = \{x_1, x_2, \ldots\}$ satisfying $x_1 \leq x_2 \leq \ldots$,

- $F_X(-\infty) = 0$ and $F_X(\infty) = 1$
- For all $x' \geq x$, $F_X(x') \geq F_X(x)$
- For $x_i \in S_X$ and small $\epsilon > 0$,

$$F_X(x_i) - F_X(x_i - \epsilon) = P_X(x_i)$$

- $F_X(x) = F_X(x_i)$ for $x_i \leq x < x_{i+1}$

Expected Value

- The expected value of X is

$$E[X] = \mu_X = \sum_{x \in S_X} x P_X(x)$$

- Also called the average of X.

Average vs. $E[X]$

- Average of n samples: $m_n = \frac{1}{n}\sum_{i=1}^{n} x(i)$
- Each $x(i) \in S_X$. If each $x \in S_X$ occurs N_x times,

$$m_n = \frac{1}{n}\sum_{x \in S_X} N_x x = \sum_{x \in S_X} \frac{N_x}{n} x \rightarrow \sum_{x \in S_X} x P_X(x)$$

Derived Random Variables

- Each sample value y of a *derived rv* Y is a function $g(x)$ of a sample value x of a rv X.
- Notation: $Y = g(X)$
- Experimental Procedure

    1. Perform experiment, observe outcome s.
    2. Find x, the value of X
    3. Calculate $y = g(x)$

PMF of $Y = g(X)$

$$P_Y(y) = \sum_{x:g(x)=y} P_X(x)$$

Problem 2.6.5

- Source transmits data packets to receiver.
- If rec'd packet is error-free, rec'vr sends back ACK, otherwise NAK sent.
- For each NAK, the packet is resent.
- Each packet transmission is independently corrupted with prob q.
- Find the PMF of X, no. of times a packet is sent
- Each packet takes 1 msec to transmit. Source waits 1 msec to receive ACK. T equal the time req'd until the packet is received OK. What is $P_T(t)$?

Expected value of $Y = g(X)$

- Thm: Given rv X with PMF $P_X(x)$, the expected value of $Y = g(X)$, is

$$E[Y] = \mu_Y = \sum_{x \in S_X} g(x) P_X(x)$$

- Example: $Y = aX + b$:

$$E[Y] = \sum_{x \in S_X} (ax + b) P_X(x) = aE[X] + b$$

Variance and Std Deviation

- Variance: $Y = (X - \mu_X)^2$

$$E[Y] = \sum_{x \in S_X} (X - \mu_X)^2 P_X(x) = \mathrm{Var}[X]$$

- Variance measures spread of PMF
- Standard Deviation: $\sigma_X = \sqrt{\mathrm{Var}[X]}$
- Units of σ_X are the same as X.

Properties of the variance

- If $Y = X + b$, $\mathrm{Var}\,[Y] = \mathrm{Var}\,[X]$.
- If $Y = aX$, $\mathrm{Var}\,[Y] = a^2\,\mathrm{Var}\,[X]$.

Conditional PMF of X given B

- Given B, with $P[B] > 0$,

$$P_{X|B}(x) = P[X = x|B]$$

- Two kinds of conditioning.

Conditional PMFs - version 1

- Probability model tells us $P_{X|B_i}(x)$ for possible B_i.
- Example: In the ith month of the year, the number of cars N crossing the GW bridge is Poisson with param α_i.

Conditional PMFs - version 2

- B is an event defined in terms of X.
- B is a subset of S_X such that for each $x \in S_X$, either $x \in B$ or $x \notin B$.

$$P_{X|B}(x) = \frac{P[X = x, B]}{P[B]} = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases}$$

Conditional PMF Example

Example: X is geometric with $p = 0.1$. What is the conditional PMF of X given event B that $X > 9$?

Conditional Expectations

- Replace $P_X(x)$ with $P_{X|B}(x|b)$
- $E[X|B] = \sum_x x P_{X|B}(x)$
- $E[g(X)|B] = \sum_x g(x) P_{X|B}(x)$
- $\mathrm{Var}[X|B] = E\left[(X - E[X|B])^2\right]$