

Probability and Stochastic Processes

A Friendly Introduction for Electrical and Computer Engineers

Chapter 3 Viewgraphs

Multiple (2) random variables

- An experiment produces both X and Y
- Example:
 - X = minutes you wait a Rutgers bus
 - Y = no. of other Rutgers buses pass by
- Joint range:

$$S_{X,Y} = \{(x, y) | P_{X,Y}(x, y) > 0\} \subset S_X \times S_Y$$

Probability Model for 2 RV's

- The *joint probability mass function* of X and Y is

$$P_{X,Y}(x, y) = P[X = x, Y = y]$$

- Joint PMF is a rule that for any x and y , gives the probability that $X = x$ and $Y = y$.

Example

- Flip a coin three times.
- $X =$ no. of H
- $Y =$ no. of T after the first H
- Write all 8 outcomes: $\{HHH, \dots, TTT\}$.
- For each outcome, record X and Y

Marginal PMF

- Experiment produces X and Y .
- All you care about is X .
- PMF of X is

$$P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x, y)$$

- Calculate by summing across rows, write PMF in margin

Functions of 2 RVs

- Derived rv $W = g(X, Y)$
- PMF of W :

$$P_W(w) = \sum_{(x,y):g(x,y)=w} P_{X,Y}(x,y)$$

Derived PMF Example

Example $W = XY$:

Expectations of Functions again

- Thm: The expected value of $W = g(X, Y)$ is

$$E[W] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y) P_{X, Y}(x, y)$$

- If $g(X, Y) = g_1(X, Y) + g_2(X, Y) + \dots + g_k(X, Y)$, then

$$E[g(X, Y)] = E[g_1(X, Y)] + \dots + E[g_n(X, Y)]$$

Expectations of Sums

- $E[X + Y] = E[X] + E[Y]$
- With $W = (X + Y - \mu_{X+Y})^2$,

$$\begin{aligned} E[W] &= \text{Var}[X + Y] \\ &= \text{Var}[X] + \text{Var}[Y] + 2E[(X - \mu_X)(Y - \mu_Y)] \end{aligned}$$

Covariance

- Covariance: $W = (X - \mu_X)(Y - \mu_Y)$,

$$\text{Cov}[X, Y] = E[W] = E[(X - \mu_X)(Y - \mu_Y)]$$

- Covariance is also

$$\text{Cov}[X, Y] = E[XY] - \mu_X \mu_Y$$

- $\text{Cov}[X, Y] > 0$ says, $X > E[X]$ implies $Y > E[Y]$ is likely (X goes up, Y goes up)
- $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}[X, Y]$

Correlation

- The *correlation* of X and Y is $E[XY]$
- Correlation = Covariance if $E[X] = E[Y] = 0$
- $E[XY] > 0$ suggests that $X > 0$ increases chance $Y > 0$
- **Orthogonal:** $E[XY] = 0$
- **Uncorrelated:** $\text{Cov}[X, Y] = 0$

Correlation Coefficient

- The *correlation coefficient* of two random variables X and Y is

$$\rho_{X,Y} = \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X] \text{Var}[Y]}}$$

- Thm: the Correlation coefficient is normalized:

$$-1 \leq \rho_{X,Y} \leq 1$$

- Thm: If $Y = aX + b$, then

$$\rho_{X,Y} = \begin{cases} -1 & a < 0 \\ 0 & a = 0 \\ 1 & a > 0 \end{cases}$$

Conditioning Events (Again!)

- The *conditional joint PMF* of X and Y given B is

$$P_{X,Y|B}(x,y) = P[X = x, Y = y|B]$$

- If B is an event on the X, Y plane,

$$\begin{aligned} P[(X = x, Y = y) \cap B] \\ = \begin{cases} P[X = x, Y = y] & (x, y) \in B \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Conditional Expectations

- The conditional expected value of $g(X, Y)$ given B is

$$E[g(X, Y) | B] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y) P_{X, Y | B}(x, y)$$

Conditional PMF of X given Y

- For any event $Y = y$ such that $P_Y(y) > 0$, the *conditional PMF of X given $Y = y$* is

$$P_{X|Y}(x|y) = P[X = x|Y = y]$$

- If $P_X(x) > 0$ and $P_Y(y) > 0$,

$$P_{X,Y}(x,y) = P_{X|Y}(x|y) P_Y(y) = P_{Y|X}(y|x) P_X(x)$$

Conditional Expected Values

- For any $y \in S_Y$, the *conditional expected value* of $g(X)$ given $Y = y$ is

$$E[g(X)|Y = y] = \sum_{x \in S_X} g(x)P_{X|Y}(x|y)$$

- Special cases
 - $E[X|Y = y]$
 - Cond. Variance:

$$\text{Var} [X|Y = y] = E \left[(X - E[X|Y = y])^2 | Y = y \right]$$

Conditional Expectation

- The conditional expectation function $E[X|Y]$ is a function of random variable Y such that if $Y = y$ then $E[X|Y] = E[X|Y = y]$
- Let $g(y) = E[X|Y = y]$. When $Y = y$, $E[X|Y]$ takes on the value $g(y)$. That is, $E[X|Y] = g(Y)$

Independent random variables

- X and Y are independent if and only if $\{X = x\}$ and $\{Y = y\}$ are independent events for all x and y
- Equivalently, X and Y are independent if

$$P_{X,Y}(x, y) = P_X(x) P_Y(y)$$

Properties of Independent RVs

- For independent random variables X and Y ,
 - $r_{X,Y} = E[XY] = E[X]E[Y]$
 - $E[X|Y = y] = E[X]$ for all $y \in S_Y$
 - $E[Y|X = x] = E[Y]$ for all $x \in S_X$
 - $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$
 - $\text{Cov}[X, Y] = \rho_{X,Y} = 0$

Multiple Discrete RVs

- The joint PMF of the discrete random variables X_1, \dots, X_n is

$$P_{X_1, \dots, X_n} (x_1, \dots, x_n) = P[X_1 = x_1, \dots, X_n = x_n]$$