$$\boxed{\text{Show } \frac{\partial \bar{E}}{\partial T} = \frac{1}{kT^2}\left(\overline{E^2} - (\bar{E})^2 \right)}$$

$$\frac{\partial \bar{E}}{\partial T} = \frac{\partial}{\partial T}\left(\frac{1}{Z} \sum_s E(s)\, e^{-E(s)/kT} \right)$$

$$= \underbrace{\left(\frac{\partial \beta}{\partial T}\right)}_{\text{use } \beta = \frac{1}{kT}} \frac{\partial}{\partial \beta}\Bigg(\underbrace{\frac{1}{Z}\sum_s E(s)\, e^{-\beta E(s)}}_{\text{use product rule}} \Bigg)$$

Z is β-dependent since $Z = \sum_s e^{-\beta E(s)}$ so you need product rule

$$= -\frac{1}{kT^2}\left\{ \left[\frac{\partial}{\partial \beta}\left(\frac{1}{Z}\right)\right]\left[\sum_s E(s)\, e^{-\beta E(s)}\right] + \left[\frac{1}{Z}\right]\left[\frac{\partial}{\partial \beta}\left(\sum_s E(s)\, e^{-\beta E(s)}\right)\right] \right\}$$

$$= -\frac{1}{kT^2}\left\{ -\frac{1}{Z^2}\left(\frac{\partial Z}{\partial \beta}\right)\left[\sum_s E(s)\, e^{-\beta E(s)}\right] + \left[\frac{1}{Z}\right]\left[\sum_s E(s) \overbrace{(-E(s))\, e^{-\beta E(s)}}^{\frac{\partial}{\partial \beta} e^{-\beta E(s)}}\right] \right\}$$

$$= -\frac{1}{kT^2}\left\{ -\frac{1}{Z^2}\left(\sum_s \frac{\partial}{\partial \beta} e^{-\beta E(s)}\right)\left[\sum_s E(s)\, e^{-\beta E(s)}\right] - \underbrace{\frac{1}{Z}\sum_s (E(s))^2 e^{-\beta E(s)}}_{\overline{E^2}} \right\}$$

$$= -\frac{1}{kT^2}\left\{ -\frac{1}{Z^2}\left(\sum_s (-E(s))\, e^{-\beta E(s)}\right)\left[\sum_s E(s)\, e^{-\beta E(s)}\right] - \overline{E^2} \right\}$$

$$= \frac{1}{kT^2}\left\{ \left(\frac{1}{Z}\sum_s E(s)\, e^{-\beta E(s)}\right)\left[\frac{1}{Z}\sum_s E(s)\, e^{-\beta E(s)}\right] - \overline{E^2} \right\}$$

$$\frac{\partial \bar{E}}{\partial T} = \frac{1}{kT^2}\left\{ (\bar{E})^2 - \overline{E^2} \right\}$$