Herbert Goldstein, "Classical Mechanics", 2nd edition, Addison Wesley

CHAPTER 10

Hamilton–Jacobi Theory

It has already been mentioned that canonical transformations may be used to provide a general procedure for solving mechanical problems. Two methods have been suggested. If the Hamiltonian is conserved then a solution could be obtained by transforming to new canonical coordinates that are all cyclic, since the integration of the new equations of motion becomes trivial. An alternative technique is to seek a canonical transformation from the coordinates and momenta, (q, p), at the time t, to a new set of constant quantities, which may be the $2n$ initial values, (q_0, p_0), at $t = 0$. With such a transformation, the equations of transformation relating the old and new canonical variables are then exactly the desired solution of the mechanical problem:

$$q = q(q_0, p_0, t),$$
$$p = p(q_0, p_0, t),$$

for they give the coordinates and momenta as a function of their initial values and the time. This last procedure is the more general one, especially as it is applicable, in principle at least, even when the Hamiltonian involves the time. We shall therefore begin our discussion by considering how such a transformation may be found.

10–1 THE HAMILTON–JACOBI EQUATION FOR HAMILTON'S PRINCIPAL FUNCTION

We can automatically ensure that the new variables are constant in time by requiring that the transformed Hamiltonian, K, shall be identically zero, for then the equations of motion are

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i = 0,$$
$$-\frac{\partial K}{\partial Q_i} = \dot{P}_i = 0. \tag{10–1}$$

As we have seen, K must be related to the old Hamiltonian and to the generating function by the equation

$$K = H + \frac{\partial F}{\partial t},$$

and hence will be zero if F satisfies the equation

$$H(q, p, t) + \frac{\partial F}{\partial t} = 0. \tag{10–2}$$

It is convenient to take F as a function of the old coordinates q_i, the new constant momenta P_i, and the time; in the notation of the previous chapter we would designate the generating function as $F_2(q, P, t)$. To write the Hamiltonian in Eq. (10–2) as a function of the same variables, use may be made of the equations of transformation (cf. Eq. 9–17a),

$$p_i = \frac{\partial F_2}{\partial q_i},$$

so that Eq. (10–2) becomes

$$H\left(q_1, \ldots, q_n; \frac{\partial F_2}{\partial q_1}, \ldots, \frac{\partial F_2}{\partial q_n}; t\right) + \frac{\partial F_2}{\partial t} = 0. \tag{10–3}$$

Equation (10–3), known as the *Hamilton–Jacobi equation*, constitutes a partial differential equation in $(n + 1)$ variables, $q_1, \ldots, q_n; t$, for the desired generating function. It is customary to denote the solution of Eq. (10–3) by S and to call it *Hamilton's principal function.*

Of course, the integration of Eq. (10–3) only provides the dependence on the old coordinates and time; it would not appear to tell how the new momenta are contained in S. Indeed the new momenta have not yet been specified except that we know they must be constants. However, the nature of the solution indicates how the new P_i's are to be selected.

Mathematically Eq. (10–3) has the form of a first-order partial differential equation in $n + 1$ variables. Suppose that there exists a solution to Eq. (10–3) of the form

$$F_2 \equiv S = S(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_{n+1}; t), \tag{10–4}$$

where the quantities $\alpha_1, \ldots, \alpha_{n+1}$ are $n + 1$ *independent* constants of integration. Such solutions are known as *complete solutions* of the first-order partial differential equation.* One of the constants of integration, however, is in fact

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* Equation (10–4) is not the only type of solution possible for Eq. (10–3). The most general form of the solution involves one or more arbitrary functions rather than arbitrary constants. See, for example, R. Courant and D. Hilbert: *Methods of Mathematical Physics*, Vol. II, 1962, pp. 24–28, and V. I. Smirnov: *A Course of Higher Mathematics*, Vol. IV, 1964, Section 111. Nor is there necessarily a unique solution of the form (10–4). There may be several complete solutions for the given equation. But all that is important for the subsequent argument is that there exist *a* complete solution.

irrelevant to the solution, for it will be noted that S itself does not appear in Eq. (10–3); only its partial derivatives with respect to q or t are involved. Hence, if S is some solution of the differential equation, then $S + \alpha$, where α is any constant, must also be a solution. One of the $n + 1$ constants of integration in Eq. (10–4) must therefore appear only as an additive constant tacked on to S. But by the same token an additive constant has no importance in a generating function, since only partial derivatives of the generating function occur in the transformation equations. Hence for our purposes a complete solution to Eq. (10–3) can be written in the form

$$S = S(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n; t), \tag{10–5}$$

where none of the n independent constants is solely additive. In this mathematical garb S tallies exactly with the desired form for an F_2 type of generating function, for Eq. (10–5) presents S as a function of n coordinates, the time t, and n independent quantities α_i. We are therefore at liberty to take the n constants of integration to be the new (constant) momenta:

$$P_i = \alpha_i. \tag{10–6}$$

Such a choice does not contradict the original assertion that the new momenta are connected with the initial values of q and p at the time t_0. The n transformation equations (9–17a) can now be written as

$$p_i = \frac{\partial S(q, \alpha, t)}{\partial q_i}, \tag{10–7}$$

where q, α stand for the complete set of quantities. At the time t_0 these constitute n equations relating the n α's with the initial q and p values, thus enabling one to evaluate the constants of integration in terms of the specific initial conditions of the problem. The other half of the equations of transformation, which provide the new constant coordinates, appear as

$$Q_i = \beta_i = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i}. \tag{10–8}$$

The constant β's can be similarly obtained from the initial conditions, simply by calculating the value of the right side of Eq. (10–8) at $t = t_0$ with the known initial values of q_i. Equations (10–8) can then be "turned inside out" to furnish q_j in terms of α, β, and t:

$$q_j = q_j(\alpha, \beta, t), \tag{10–9}$$

which solves the problem of giving the coordinates as functions of time and the initial conditions.* After the differentiation in Eqs. (10–7) has been performed, Eqs. (10–9) may be substituted for the q's, thus giving the momenta p_i as functions of the α, β, and t:

$$p_i = p_i(\alpha, \beta, t). \qquad (10\text{–}10)$$

Equations (10–9) and (10–10) thus constitute the desired complete solution of Hamilton's equations of motion.

Hamilton's principal function is thus the generator of a canonical transformation to constant coordinates and momenta; *when solving the Hamilton–Jacobi equation we are at the same time obtaining a solution to the mechanical problem.* Mathematically speaking, we have established an equivalence between the $2n$ canonical equations of motion, which are first-order differential equations, to the first-order partial differential Hamilton–Jacobi equation. This correspondence is not restricted to equations governed by the Hamiltonian; indeed the general theory of first-order partial differential equations is largely concerned with the properties of the equivalent set of first-order ordinary differential equations. Essentially, the connection can be traced to the fact that both the partial differential equation and its canonical equations stem from a common variational principle, in this case Hamilton's modified principle.

To a certain extent the choice of the α_i's as the new momenta is arbitrary. One could just as well choose any n quantities, γ_i, which are independent functions of the α_i constants of integration:

$$\gamma_i = \gamma_i(\alpha_1, \ldots, \alpha_n). \qquad (10\text{–}11)$$

By means of these defining relations Hamilton's principal function can be written as a function of q_i, γ_i, and t, and the rest of the derivation then goes through unchanged. It often proves convenient to take some particular set of γ_i's as the new momenta, rather than the constants of integration that appear naturally in integrating the Hamilton–Jacobi equation.

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* As a mathematical point it may be questioned whether the process of "turning inside out" is feasible for Eqs (10–7) and (10–8), i.e., whether they can be solved for α_i and q_i respectively. The question hinges on whether the equations in each set are independent, for otherwise they are obviously not sufficient to determine the n independent quantities α_i or q_i as the case may be. To simplify the notation, let S_α symbolize members of the set of partial derivatives of S with respect to α_i, so that Eq. (10–8) is represented by $\beta = S_\alpha$. That the derivatives S_α in (10–8) form independent functions of the q's follows directly from the nature of a complete solution to the Hamilton–Jacobi equation; indeed this is what we mean by saying the n constants of integration are independent. Consequently the Jacobian of S_α with respect to q_i cannot vanish. Since the order of differentiation is immaterial, this is equivalent to saying that the Jacobian of S_q with respect to α_i cannot vanish, which proves the independence of Eqs. (10–7).

Further insight into the physical significance of S is furnished by an examination of its total time derivative, which can be computed from the formula

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t},$$

since the P_i's are constant in time. By Eqs. (10–7) and (10–3) this relation can also be written

$$\frac{dS}{dt} = p_i\dot{q}_i - H = L, \tag{10–12}$$

so that Hamilton's principal function differs at most from the indefinite time integral of the Lagrangian only by a constant:

$$S = \int L\,dt + \text{constant}. \tag{10–13}$$

Now, Hamilton's principle is a statement about the definite integral of L, and from it we obtained the solution of the problem via the Lagrange equations. Here the same action integral, in an indefinite form, furnishes another way of solving the problem. In actual calculations the result expressed by Eq. (10–13) is of no help, because one cannot integrate the Lagrangian with respect to time until q_i and p_i are known as functions of time, i.e., until the problem is solved.*

10–2 THE HARMONIC OSCILLATOR PROBLEM AS AN EXAMPLE OF THE HAMILTON–JACOBI METHOD

To illustrate the Hamilton–Jacobi technique for solving the motion of mechanical systems we shall work out in detail the simple problem of a one-dimensional harmonic oscillator. The Hamiltonian is

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \equiv E, \tag{10–14}$$

where

$$\omega = \sqrt{\frac{k}{m}}, \tag{10–15}$$

* Historically the recognition by Hamilton that the time integral of L is a special solution of a partial differential equation came before it was seen how the Hamilton–Jacobi equation can furnish the solution to a mechanical problem. It was Jacobi who realized that the converse was true, that by the techniques of canonical transformations any complete solution of the Hamiltonian–Jacobi equation could be used to describe the motion of the system.