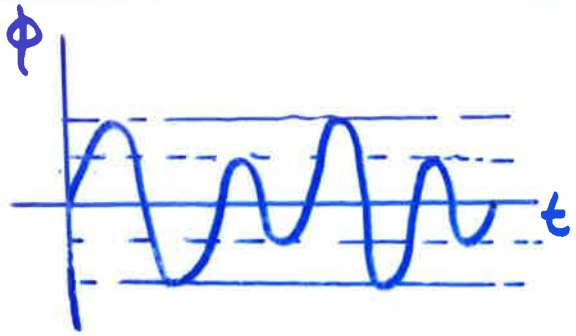
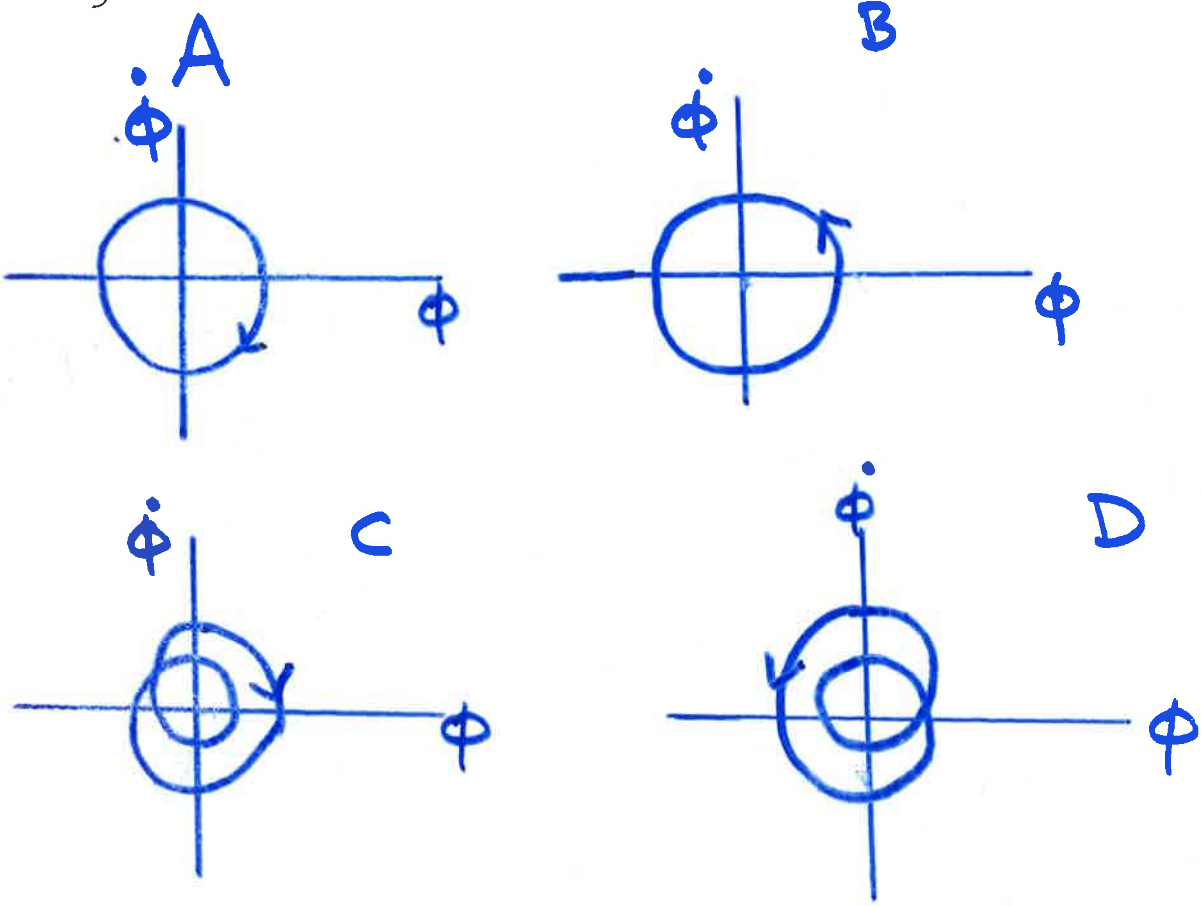
Newton (1coin)



Which $\dot{\phi}(\phi)$ corresponds to this $\phi(t)$? A, B, C or D?



C

Newton (2 coins)

particles $i = 1, 2, \ldots, N$

Central forces $\underline{F}_{ij}$, no other forces

Newton's laws valid

Show $\underline{L}^{tot} = \text{const.}$ (vector symbols incl.)

$$\frac{d}{dt}\underline{L}^{tot} = \frac{d}{dt}\sum_{i=1}^{N} \underline{r}_i \times m_i \underline{v}_i = \overbrace{\sum_i \dot{\underline{r}}_i \times m_i \dot{\underline{r}}_i}^{=0} + \sum_i \underline{r}_i \times m_i \ddot{\underline{r}}_i$$

$$\underset{\substack{\uparrow \\ N2^{nd}}}{=} \sum_i \cancel{m_i}\, \underline{r}_i \times \underline{F}_i \underset{\substack{\uparrow \\ \text{no other} \\ \text{forces}}}{=} \sum_i \sum_j m_i\, \underline{r}_i \times \underline{F}_{ij}$$

$$= \sum_{i>j} \cancel{m_i} \left(\underline{r}_i \times \underline{F}_{ij} + \underline{r}_j \times \underline{F}_{ji} \right)$$

$$\underset{\substack{\uparrow \\ N3^{rd}}}{=} \sum_{i>j} \cancel{m_i} \left(\underline{r}_i - \underline{r}_j \right) \times \underline{F}_{ij}$$

$$\underset{\substack{\uparrow \\ \text{central} \\ \text{forces}}}{=} 0$$

Newton (3 coins)

$$U = \frac{kq^2}{r_{12}} - \frac{2kq^2}{r_{13}} - \frac{2kq^2}{r_{23}}$$

$$r_{ij} = |\underline{r}_i - \underline{r}_j| \qquad k = \text{const.}$$

Determine $\underline{F}_1$.

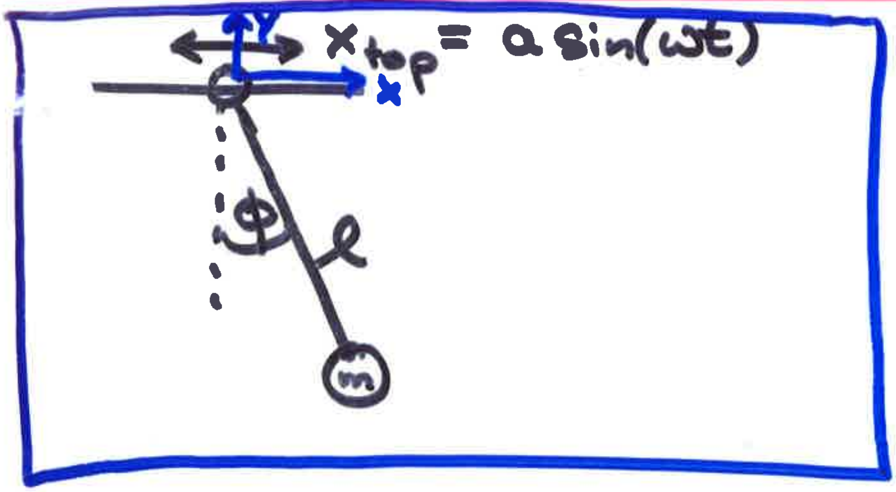
$$\underline{F}_1 = -\underline{\nabla}_1 U = -\underline{\nabla}_1 (U_{12} + U_{13})$$

$$F_{1x} = -\frac{\partial}{\partial x_1}\left[\frac{kq^2}{r_{12}} - \frac{2kq^2}{r_{13}}\right] = \underbrace{-\frac{\partial}{\partial x_1}\left[\frac{kq^2}{r_{12}}\right]}_{+\frac{kq^2}{r_{12}^2}}\frac{\partial r_{12}}{\partial x_1} \; \underbrace{+\frac{\partial}{\partial x_1}\left[\frac{2kq^2}{r_{13}}\right]}_{-\frac{2kq^2}{r_{13}^2}}\frac{\partial r_{13}}{\partial x_1}$$

$$\frac{\partial r_{12}}{\partial x_1} = \frac{\partial}{\partial x_1}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + \ldots} = +\frac{2(x_1 - x_2)}{2\sqrt{\cdot}}$$

$$\underline{F}_1 = \frac{kq^2}{r_{12}^3}(\underline{r}_1 - \underline{r}_2) - \frac{2kq^2}{r_{13}^3}(\underline{r}_1 - \underline{r}_3)$$

Lagrange etc. (1coin + 1coin + 2coins)



a) $\mathcal{L} = \mathcal{L}(x,y) = ?$ [1coin]

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - (+mgy)$$

b) $x = x(\phi) = ?$ $y = y(\phi) = ?$ [1coin]

$$\begin{cases} x = x_{top} + \ell\sin\phi = a\sin(\omega t) + \ell\sin\phi \\ y = -\ell\cos\phi \end{cases}$$

c) $\mathcal{L} = \mathcal{L}(\phi, \dot{\phi}) = ?$ [2coins]

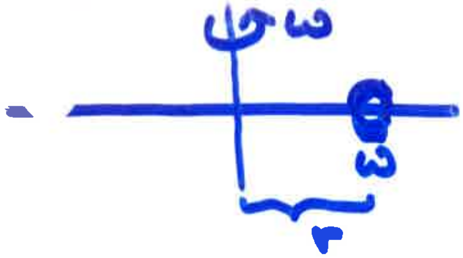
$\dot{x} = a\omega\cos(\omega t) + \ell\dot{\phi}\cos\phi$ $\dot{y} = \ell\dot{\phi}\sin\phi$

$$\mathcal{L} = \frac{m}{2}\left[(a\omega\cos(\omega t) + \ell\dot{\phi}\cos\phi)^2 + (\ell\dot{\phi}\sin\phi)^2\right] + mg\ell\cos\phi$$

$$= \frac{m}{2}\left[a^2\omega^2\cos^2(\omega t) + 2a\omega\cos(\omega t)\ell\dot{\phi}\cos\phi + \ell^2\dot{\phi}^2\right]$$
$$+ mg\ell\cos\phi$$

d) $\frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}$ (not today)

Lagrange etc. (Hamilton) [2coins + 2coins + 1coin]



bead on straight frictionless rod, horizontal

a) $\mathcal{L} = \mathcal{L}(r, \dot{r}) = ?$ [2coins]

$x = r\cos\phi = r\cos(\omega t)$ $\qquad$ $\dot{x} = \dot{r}\cos(\omega t) - r\omega\sin(\omega t)$

$y = r\sin(\omega t)$ $\qquad$ $\dot{y} = \dot{r}\sin(\omega t) + r\omega\cos(\omega t)$

$$\mathcal{L} = T - U = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{m}{2}\left(\dot{r}^2 + r^2\omega^2\right)$$

b) $\mathcal{H} = \mathcal{H}(r, p) = ?$ [2coins]

$$p = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \rightarrow \dot{r} = \frac{p}{m}$$

$$\mathcal{H} = p\dot{r} - \mathcal{L} = \frac{p^2}{m} - \frac{p^2}{2m} - \frac{m}{2}r^2\omega^2 = \frac{p^2}{2m} - \frac{m}{2}r^2\omega^2$$

Note: $\mathcal{H} \neq T+U$

c) Hamilton's equations [1coin]

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial r} = m\omega^2 r$$

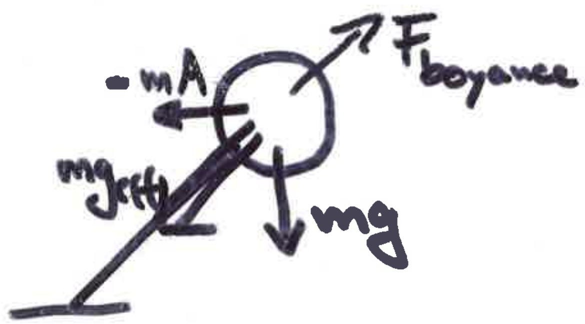
Rotations etc. (1 coin)

? Helium balloon is anchored by a massless string to the floor of a car that is accelerated forward. In the frame of the car the balloon

① tilting forward

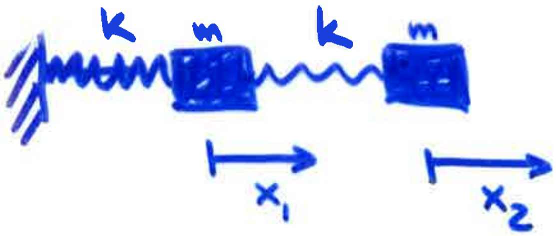
② straight up

③ tilting backwards



$\Rightarrow$ ①

Rotations etc. (3 coins)



Determine eigenfrequencies

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -2k\,x_1 + k\,x_2$$

$$m\ddot{x}_2 = -k(x_2 - x_1)$$

$$\underline{\underline{M}} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad \underline{\underline{K}} = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}$$

$$\underline{\underline{M}}\,\ddot{\underline{x}} = -\underline{\underline{K}}\,\underline{x} \qquad \underline{x} = \mathrm{Re}\,\underline{a}\,e^{i\omega t}$$

$$\det(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) = \det\begin{pmatrix} (2k-\omega^2 m) & -k \\ -k & (k-\omega^2 m) \end{pmatrix}$$

$$= (2k - \omega^2 m)(k - \omega^2 m) - k^2$$

$$= 2k^2 - 3k\omega^2 m + \omega^4 m^2 - k^2$$

$$\omega^4 - 3\frac{k}{m}\omega^2 + \left(\frac{k}{m}\right)^2 = 0$$

$$\omega^2 = \left(\frac{3}{2} \pm \sqrt{\frac{9}{4} - 1}\right)\frac{k}{m}$$

$$\omega = \sqrt{\frac{3 \pm \sqrt{5}}{2}}\sqrt{\frac{k}{m}}$$