Homework #33

Problem HA: Direct Approach

Similar to the derivation in class, derive the direct conditions

$$\left(\frac{\partial P_i}{\partial p_j}\right)_{\mathbf{p},\mathbf{q}} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{\mathbf{Q},\mathbf{P}} \qquad \left(\frac{\partial P_i}{\partial q_j}\right)_{\mathbf{p},\mathbf{q}} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{\mathbf{Q},\mathbf{P}}$$

Problem HB

In Problem 13.25 you had

$$q(Q,P) = \sqrt{2P}\sin Q \qquad p(Q,P) = \sqrt{2P}\cos Q$$

This is an example of a restricted canonical transformation, where no time dependence appears in the mapping.

(a) Solve for $Q(q,p)$ and $P(q,p)$.

(b) Verify that this transformation is a restricted canonical transformation by showing it satisfies the four direct conditions:

$$\left(\frac{\partial Q}{\partial p}\right)_q = -\left(\frac{\partial q}{\partial P}\right)_Q \qquad \left(\frac{\partial Q}{\partial q}\right)_p = \left(\frac{\partial p}{\partial P}\right)_Q$$
$$\left(\frac{\partial P}{\partial p}\right)_q = \left(\frac{\partial q}{\partial Q}\right)_P \qquad \left(\frac{\partial P}{\partial q}\right)_p = -\left(\frac{\partial p}{\partial Q}\right)_P$$

(c) The space of possible canonical transformations is vast. It is helpful to have some general techniques for generating them. Here is one.[1] Let $\phi(q,Q)$ be any function of q and Q. This ϕ is called a *generating function* because if we set

$$p = \frac{\partial \phi}{\partial q} \qquad P = -\frac{\partial \phi}{\partial Q}$$

this defines a relation between (q,p) and (Q,P) that is a canonical transformation. For any choice of $\phi(q,Q)$! Show that the choice

$$\phi(q,Q) = \tfrac{1}{2}q^2\cot Q$$

leads to the canonical transformation of Problem 13.25.

(d) This is a quickie: find the canonical transformation corresponding to $\phi(q,Q) = qQ$.

[1] There is a variational method, like Hamilton's principle, that explains why this generating function technique works. Check which generating function type this is.