Separation of Variables

Classes Sept. 25 & 28
HW17 & HW18
Book §3.3

Cartesian Coordinates

$$V(x,y,z) = X(x)\,Y(y)\,Z(z)$$

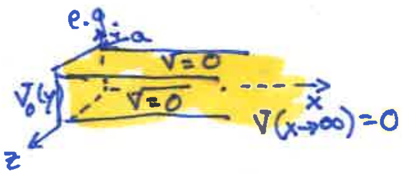
$$\vec{\nabla}^2 V = 0$$

e.g. z-indep

$$V(x,y) = \left(A e^{kx} + B e^{-kx}\right)\left(C\sin(ky) + D\cos(ky)\right)$$

or swapped

Boundary Conditions:



e.g. $k = n\frac{\pi}{a}$

e.g. $V(x,y) = \sum_{n=1}^{\infty} C_n\, e^{-n\pi x/a} \sin\left(\frac{n\pi}{a}y\right)$

Spherical Coordinates

$$V(r,\phi,\theta) = R(r)\,\Phi(\phi)\,\Theta(\theta)$$

$$\vec{\nabla}^2 V = 0$$

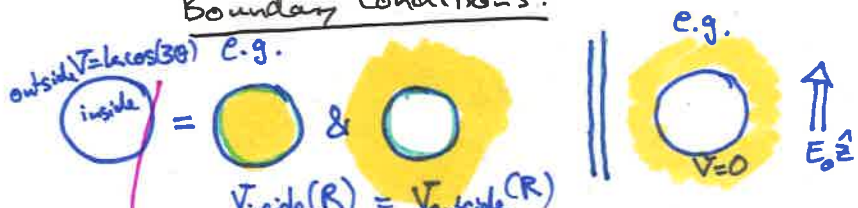
e.g. azimuthal symmetry
ϕ indep.

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell \frac{1}{r^{(\ell+1)}}\right) P_\ell(\cos\theta)$$

Legendre Polynomials

$P_0(x)=1$ $P_1(x)=x$ $P_2(x)=(3x^2-1)\frac{1}{2}$
$P_3(x) = (5x^3-3x)\frac{1}{2}$

Boundary Conditions:



$V_{inside}(R) = V_{outside}(R)$

$V(r\to 0) \neq \infty$
$\hookrightarrow B_\ell = 0$

$V(r\to\infty) = 0$
$\hookrightarrow A_\ell = 0$

Cylindrical Coordinates

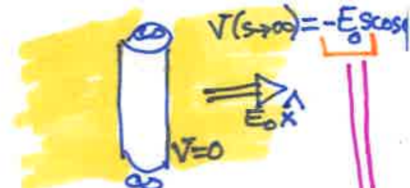
$$V(s,\phi,z) = S(s)\,\Phi(\phi)\,Z(z)$$

$$\vec{\nabla}^2 V = 0$$

e.g. z-indep.

$$V(s,\phi) = a_0 + b_0 \ln s + \sum_{k=1}^{\infty} s^k\left(a_k\cos(k\phi) + b_k\sin(k\phi)\right) + \sum_{k=1}^{\infty} s^{-k}\left(c_k\cos(k\phi) + d_k\sin(k\phi)\right)$$

Boundary Conditions



Determine coefficient of remaining series:

e.g.

$$V_0 = \sum_{n=1}^{\infty} C_n \underbrace{e^{0}}_{=1} \sin\left(\frac{n\pi}{a}y\right)$$

Fourier's Trick: Use $\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = \begin{cases} 0 & n\neq n' \\ a/2 & n=n' \end{cases}$

$$C_{n'} = \frac{2}{a}\int_0^a V_0 \sin\left(\frac{n'\pi}{a}y\right)dy$$

$$V_0(\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta)$$

Use $\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,dx = \int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \begin{cases} 0 & \text{for } \ell\neq\ell' \\ \frac{2}{(2\ell+1)} & \text{for } \ell=\ell' \end{cases}$

$$A_{\ell'} = \frac{(2\ell'+1)}{2}\int_0^\pi V_0(\theta)\,P_{\ell'}(\cos\theta)\sin\theta\, d\theta$$

Compare Coefficients Directly:

$k\cos(3\theta)$
$= k\left[4\cos^3\theta - 3\cos\theta\right] = k\left[\alpha P_3(\cos\theta) + \beta P_1(\cos\theta)\right]$
$= \dots = k\frac{\alpha}{2}5\cos^3\theta + \left(-3\frac{\alpha}{2}+\beta\right)\cos\theta$

$a_k, b_k, c_k, d_k = 0$
for all $k\neq 1$

$V(s,\phi) = s a_1\cos(\phi) + s^{-1}c_1\cos(\phi)$

$V(s\to\infty,\phi) = s a_1 \cos\phi = -E_0 s\cos\phi$

Compare Coefficients Directly

Basisvectors

example:

$$6\boxed{\hat{x}} + 2\boxed{\hat{y}} = A_1(\hat{x}+3\hat{y}) + B_1(\hat{x}+2\hat{y})$$

$$= \underbrace{(A_1+B_1)}\boxed{\hat{x}} + \underbrace{(3A_1+2B_1)}\boxed{\hat{y}}$$

determine

polynomials of $\cos\theta$

$$k\cos(3\theta)$$

$$= \underbrace{k \cdot 4\boxed{\cos^3\theta} - 3k\boxed{\cos\theta}}_{\text{given as boundary condition}} = \sum_{\ell=0}^{\infty} A_\ell R^\ell \overbrace{P_\ell(\cos\theta)}$$

$$= A_0\boxed{1} + A_1 R_1 \boxed{\cos\theta} + A_2 R^2 (3\boxed{\cos^2\theta} - 1)\tfrac{1}{2} + A_3 R^3 (5\boxed{\cos^3\theta} - 3\boxed{\cos\theta})\tfrac{1}{2} + \ldots$$

$\rightarrow$ only $A_1 R^1 P_1(\cos\theta)$ and $A_3 R^3 P_3(\cos\theta)$

$$\ldots = \underbrace{(A_1 R - \tfrac{3}{2}A_3 R^3)}\boxed{\cos\theta} + \underbrace{\tfrac{5}{2}A_3 R^3}\boxed{\cos^3\theta}$$

Fourier's Trick

Basis vectors e.g. $\hat{a} = \begin{pmatrix}1\\1\\1\end{pmatrix}\frac{1}{\sqrt{3}}$ $\hat{b} = \begin{pmatrix}1\\-1\\0\end{pmatrix}\frac{1}{\sqrt{2}}$ $\hat{c} = \begin{pmatrix}1\\1\\-2\end{pmatrix}\frac{1}{\sqrt{6}}$

notice $\hat{a}\cdot\hat{b}=0$
$\hat{a}\cdot\hat{c}=0$
$\hat{b}\cdot\hat{c}=0$
$\hat{a}\cdot\hat{a}=1$
$\hat{b}\cdot\hat{b}=1$
$\hat{c}\cdot\hat{c}=1$

$$6\hat{x} = C_1\boxed{\hat{a}} + C_2\boxed{\hat{b}} + C_3\boxed{\hat{c}}$$

Basisvectors

$$6\hat{x}\boxed{\cdot\,\hat{a}} = C_1$$

$$6\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\frac{1}{\sqrt{3}} = \frac{6}{\sqrt{3}} = C_1$$

$$6\hat{x}\cdot\hat{b} = C_2 \qquad \text{etc.}$$

$$V_0 = \sum_{n=1}^{\infty} C_n \boxed{\sin\left(\frac{n\pi}{a}y\right)}$$

$$C_{n'} = \boxed{\frac{2}{a}\int_0^a} V_0 \boxed{\sin\left(\frac{n'\pi}{a}y\right)dy}$$