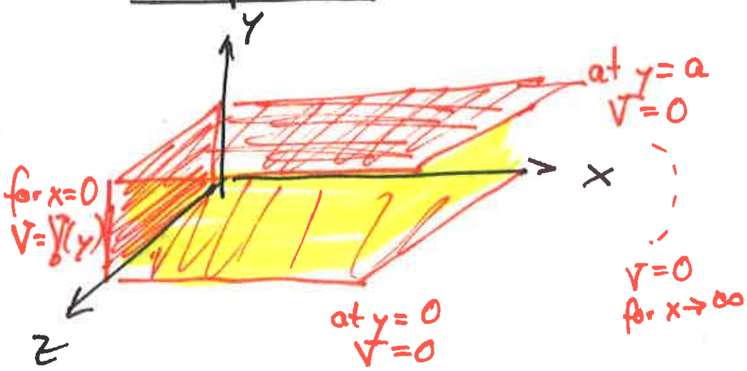
Separation of Variables

Example 3.3



I $V(x,y,z) = X(x)\,Y(y)\,Z(z)$

II $\nabla^2 V \underset{\text{cartes. coord.}}{=} \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

Apply symmetry: here boundary conditions are z-indep.

$\rightarrow \frac{\partial^2 V}{\partial z^2} = 0$ and $Z(z) = \text{const.}$

$$\nabla^2 V \underset{I}{=} Y\frac{\partial^2 V}{\partial x^2} + X\frac{\partial^2 V}{\partial y^2} = 0 \quad \Big\downarrow \frac{1}{XY}$$

$$\frac{1}{X}\frac{\partial^2 V}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 V}{\partial y^2}$$

III $= \text{const.}$

$$\frac{\partial^2 X}{\partial x^2} = k^2 X \rightarrow X = Ae^{kx} + Be^{-kx}$$

$$\frac{\partial^2 Y}{\partial y^2} = -k^2 Y \rightarrow Y = C\sin(ky) + D\sin(ky)$$

OR x,y swapped

$$V(x,y,z) = (Ae^{kx} + Be^{-kx})(C\sin(ky) + D\cos(ky))$$

IV
(i) $V = 0$ for $y = 0$
(ii) $V = 0$ for $y = a$
(iii) $V = V_0(y)$ for $x = 0$
(iv) $V \rightarrow 0$ for $x \rightarrow \infty$

General Approach

- Make Sketch (boundary conditions, yellow region where we want to find solution

I Assume Product

Ia cartesian $V(x,y,z) = X(x)Y(y)Z(z)$

Ib spherical $V(r,\theta,\phi) = R(r)\Phi(\phi)\Theta(\theta)$

Ic cylindrical HW17

II $\nabla^2 V = 0$

IIa cartes. coord.
IIb. spherical coord.
IIc cylindrical coord

see back of book; HW17

& Apply Symmetry

spherical ϕ-indep.
cylindr. z-indep

& Plug V into $\nabla^2 V = 0$ and separate variables → const.

III General Solution

IIIa $(Ae^{kx} + Be^{-kx})(C\sin(ky) + D\cos(ky))$ OR x,y swapped

IIIb $V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$

IIIc HW17

IV Name Boundary Conditions

$$V(x,y,z) = (Ae^{kx} + Be^{-kx})(C\sin(ky) + D\cos(ky))$$

correct here that x (not x-y swapped)

V $V(x \to \infty) \overset{(iv)}{=} 0 \;\longrightarrow\; A = 0$

V Apply Boundary Conditions

$$\longrightarrow\; V(x,y) = Be^{-kx}(C\sin(ky) + D\cos(ky))$$

↓ BC is const, call C and DB is const, call D

Va decide whether x or y get $(e^{kx} + e^{-kx})$

start with $V=0$ boundary conditions

next: $V(x, y=0) = 0$

$$V(x, y=0) = e^{-kx}(C\sin(0) + D\cos(0))$$

$$= e^{-kx} D = 0 \quad \text{for all } x > 0$$

$$\hookrightarrow D = 0$$

$$\longrightarrow\; V(x,y) = C e^{-kx}\sin(ky)$$

next: $V(x, y=a) = 0 \;\overset{!}{=}\; C e^{-kx}\sin(ka) = 0$

$$\rightarrow ka = n\pi \qquad n = 1, 2, 3, 4, \ldots$$

$$k = \frac{n\pi}{a}$$

So most general solution

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi}{a}y\right)$$

do toughest boundary condition last

(Fourier's Trick or direct comparison)

next: $V(x=0) = V_0(y) = \sum_{n=1}^{\infty} C_n \underbrace{e^{0}}_{=1} \sin\left(\frac{n\pi}{a}y\right)$

Direct Read off of coefficients possible if:

e.g. $V_0(y) = \underbrace{3.1}_{C_5 = 3.1 \text{ coefficient}} \underbrace{\sin\left(\frac{5\pi}{a}y\right)}_{n=5 \text{ basis vector}} + \underbrace{7.6}_{C_9 = 7.6 \text{ coefficient}} \underbrace{\sin\left(\frac{9\pi}{a}y\right)}_{n=9 \text{ basis vector}}$

OR if $V_0(y)$ more general:

multiply both sides with $\sin\left(\frac{n'\pi}{a}y\right)$ and integrate $\int \ldots dy$

Scalarproduct equivalent

$$\underbrace{C_n}_{\text{coefficient}} = \frac{2}{a}\int_0^a V_0(y) \underbrace{\sin\left(n\frac{\pi y}{a}\right)}_{\text{basis vector}} dy$$

do for all n and put back into $V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi}{a}y\right)$

e.g. for $V(x=0,y) = V_0$

Special case of $V_0(y) = V_0 = \text{const.}$

$$C_n = \frac{2}{a} V_0 \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy = \frac{2V_0}{a} \left. \frac{\left(-\cos\left(\frac{n\pi}{a}y\right)\right)}{\frac{n\pi}{a}} \right|_0^a$$

$$= \frac{2V_0}{n\pi}\Big(\underbrace{-\cos(n\pi)}_{\substack{\text{if } n \text{ even } -1 \\ \text{if } n \text{ odd } +1}} + \underbrace{\cos(0)}_{=1}\Big) = \begin{cases} \frac{2V_0}{n\pi}(-1+1) = 0 & \text{if } n \text{ even} \\ \frac{4V_0}{n\pi} & \text{if } n \text{ odd} \end{cases}$$

VI Coefficients Back into General Solution

back into solution:

$$V(x,y) = \sum_{n=1}^{\infty} C_n' \, e^{-n\pi x/a} \sin\left(\frac{n\pi}{a}y\right)$$

$$\boxed{V(x,y) = \sum_{n=1,3,5,\dots} \frac{4V_0}{n\pi} e^{-n\pi x/a} \sin\left(\frac{n\pi}{a}y\right)}$$

VII Sometimes

$$\sigma = -\varepsilon\left(\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n}\right)\Bigg|_{\text{at surface}}$$

Sometimes

$$\vec{F} = q\vec{E} = -q\vec{\nabla}V$$