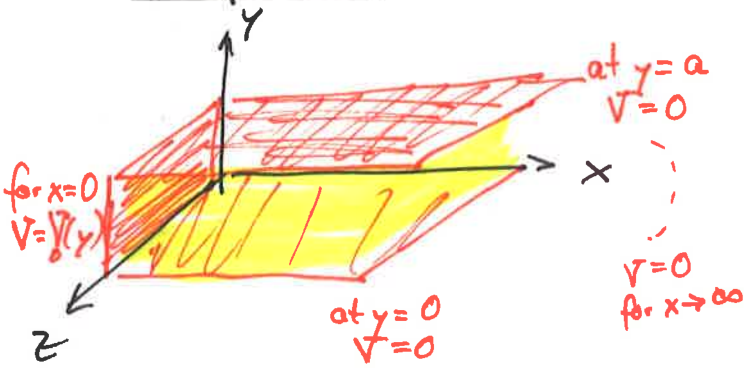
Separation of Variables

Example 3.3



I $V(x,y,z) = X(x)\,Y(y)\,Z(z)$

II $\underset{\text{Cartes. Coord.}}{\nabla^2 V =} \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$

Apply symmetry: here boundary conditions are z-indep.

$\rightarrow \dfrac{\partial^2 V}{\partial z^2} = 0$ and $Z(z) = \text{const.}$

$$\nabla^2 V \underset{\text{I}}{=} Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0 \quad \Big\downarrow \frac{1}{XY}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

III $\qquad = \text{const.}$

$$\frac{\partial^2 X}{\partial x^2} = k^2 X \rightarrow X = A e^{kx} + B e^{-kx}$$

$$\frac{\partial^2 Y}{\partial y^2} = -k^2 Y \rightarrow Y = C \sin(ky) + D \sin(ky)$$

OR x,y swapped

$$V(x,y,z) = (Ae^{kx} + Be^{-kx})(C\sin(ky) + D\cos(ky))$$

IV
(i) $V = 0$ for $y = 0$
(ii) $V = 0$ for $y = a$
(iii) $V = V_0(y)$ for $x = 0$
(iv) $V \rightarrow 0$ for $x \rightarrow \infty$

General Approach

- Make Sketch (boundary conditions, yellow region where we want to find solution)

I Assume Product

Ia cartesian $V(x,y,z) = X(x)Y(y)Z(z)$

Ib spherical $V(r,\theta,\phi) = R(r)\Phi(\phi)\Theta(\theta)$

Ic cylindrical HW17

II $\nabla^2 V = 0$

IIa cartes. coord.
IIb. spherical coord.
IIc cylindrical coord
} see back of book HW17

& Apply Symmetry

spherical ϕ-indep.
cylindr. z-indep

& Plug V into $\nabla^2 V = 0$ and separate variables → const.

III General Solution

IIIa $(Ae^{kx} + Be^{-kx})(C\sin(ky) + D\cos(ky))$ OR x,y swapped

IIIb $V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \dfrac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

IIIc HW 17

IV Name Boundary Conditions