Separation of Variables

Book §3.3

Cartesian Coordinates

$$V(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$\vec{\nabla}^2 V = 0$$

↓ e.g. z-indep

$$V(x,y) = \left(A e^{kx} + B e^{-kx}\right)\left(C\sin(ky) + D\cos(ky)\right)$$

or swapped

Spherical Coordinates

$$V(r,\phi,\theta) = R(r)\,\Phi(\phi)\,\Theta(\theta)$$

$$\vec{\nabla}^2 V = 0$$

e.g. azimuthal symmetry, ϕ indep.

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell \frac{1}{r^{(\ell+1)}}\right) P_\ell(\cos\theta)$$

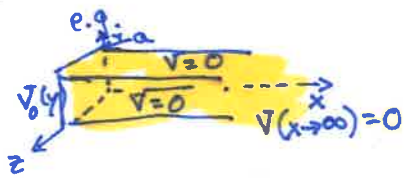
Legendre Polynomials

$P_0(x)=1$ $\quad P_1(x) = x$ $\quad P_2(x) = (3x^2-1)\frac{1}{2}$

$P_3(x) = (5x^3 - 3x)\frac{1}{2}$

Boundary Conditions:

e.g.



e.g. $k = n\frac{\pi}{a}$

e.g. $V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi}{a}y\right)$

Boundary Conditions:

e.g.

$V_{inside}(R) = V_{outside}(R)$

$V(r\to 0) \neq \infty$ $\quad V(r\to\infty) = 0$

$\hookrightarrow B_\ell = 0$ $\quad \hookrightarrow A_\ell = 0$

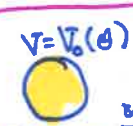
Determine coefficient of remaining series:

e.g.

$$V_0 = \sum_{n=1}^{\infty} C_n \underbrace{e^{0}}_{=1} \sin\left(\frac{n\pi}{a}y\right)$$

Fourier's Trick: Use $\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = \begin{cases} 0 & n\neq n' \\ a/2 & n=n' \end{cases}$

$$C_{n'} = \frac{2}{a}\int_0^a V_0 \sin\left(\frac{n'\pi}{a}y\right)dy$$



$$V_0(\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta)$$

Use $\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\,dx = \int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\,d\theta = \begin{cases} 0 & \text{for } \ell\neq\ell' \\ \frac{2}{(2\ell+1)} & \text{for } \ell=\ell' \end{cases}$

$$A_{\ell'} = \frac{(2\ell'+1)}{2}\int_0^\pi V_0(\theta)\,P_{\ell'}(\cos\theta)\sin\theta\,d\theta$$

Compare Coefficients Directly:

$$k\cos(3\theta) = k\left[4\cos^3\theta - 3\cos\theta\right] = k\left[\alpha P_3(\cos\theta) + \beta P_1(\cos\theta)\right]$$

$$= \ldots = k\frac{\alpha}{2}5\cos^3\theta + \left(-\frac{3\alpha}{2}+\beta\right)\cos\theta$$