

SIEGEL MODULAR FORMS IN SAGE

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A classical modular form is a periodic function on the complex upper half-plane whose Fourier coefficients encode a wealth of number theoretic information. In computer algebra systems such as Sage [9] and MAGMA [2] a generic classical modular form is computed via modular symbols and these implementations have been used in a great deal of research. For modular forms for the full modular group $SL(2, \mathbb{Z})$, Eisenstein series and the cusp form of weight 12 have their own highly optimized implementations in Sage.

A Siegel modular form of genus n is, roughly speaking, a modular form in $n(n+1)/2$ variables; in particular a Siegel modular form of genus 1 is a classical modular form. Code co-written by Craig Citro, Alex Ghitza, Nils-Peter Skoruppa and the author provides an implementation of the ring of Siegel modular forms of genus two (the code is based on [10]). In [4] it was proved that the ring of even weight Siegel modular forms on the full modular group $Sp(4, \mathbb{Z})$ of four by four symplectic matrices was generated by four forms: an Eisenstein series of weight four, one of weight six, and two cusp forms, one of weight 10 and the other of weight 12. In [3], these generators are efficiently computed:

```
A,B,C,D = siegel_modular_forms_generators( prec = 501)
```

The Fourier expansion of a Siegel modular form is indexed by positive semi-definite matrices with integer coefficients. Each matrix index should be thought of as a quadratic form and the precision of a form is the discriminant of the largest quadratic form in the support of the form.

The coefficients of a form are stored as a dictionary: the keys are triples $[a, b, c]$ corresponding to the quadratic form $aX^2 + bXY + cY^2$. If two quadratic forms are $GL(2, \mathbb{Z})$ equivalent, the corresponding Fourier coefficients are the same and so we only keep those keys that are $GL(2, \mathbb{Z})$ -reduced.

The Siegel modular form class implementation includes, among other methods, the scalar multiplication, addition and multiplication of forms, the action of Hecke operators on a given modular form, and an implementation of the Saito-Kurokawa lift. As an intermediate step in the computation of a Sato-Kurokawa lift, we construct a pared-down Jacobi form. Of particular interest is the use of Cython (a compiled Python-like language [1]) for the multiplication of two Siegel modular forms.

For example, for a Siegel modular form S that is an eigenform with respect to the Hecke operators and an integer N we can compute

```
T = HeckeOperator(N)
lambda_N = T(S)[(1,1,1)]/S[(1,1,1)]
```

The value `lambda_N`, then, is the Hecke eigenvalue of S with respect to the Hecke operator $T(N)$. The code `S[(1,1,1)]` returns the coefficient of S indexed by the quadratic form $X^2 + XY + Y^2$.

The code thus far has been used (or is being used) in the following projects:

- the formulation and analysis of the Sato-Tate conjecture for Siegel modular forms [6];
- the computation of the weight 35 Siegel modular form [7]; and
- the investigation of a Dirichlet series [8], first defined in [5], attached to a Siegel modular form.

There are also a number of other projects in their formative stages in which this code will be used.

As Siegel modular forms are multivariate power series, it is somewhat expensive to compute them. In the future, we intend to pre-compute a large number of coefficients of all the Siegel modular forms of degree two that are Hecke eigenforms and have weight less than 100. By serializing the resulting Sage objects and posting them on an online database, other researchers will be able to use the data to formulate and test conjectures and to discover previously unobserved phenomena.

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