Computer Methods for Siegel Modular Forms

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Ingredients

- Arithmetic group
- Upper half-plane
- Automorphy factor
- Growth condition
Ingredients

- Arithmetic group: finite index subgroup

$$\Gamma^n \subset Sp_{2n}(\mathbb{Z}) = \{ M \in GL_{2n}(\mathbb{Z}) : M^tJM = J \}, \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

- Upper half-plane: $$\mathcal{H}_n = \{ Z \in M_n(\mathbb{C}) : Z^t = Z, \text{Im}(Z) > 0 \}.$$

- Let $$\rho : GL(n, \mathbb{C}) \rightarrow GL(V)$$ be a rational representation of a finite dimensional vector space.
Let $M^\rho_n(\Gamma^n) = M^n_\rho$ be the space of Siegel modular forms of weight $\rho$ and degree $n$. I.e., $F \in M^n_\rho$ iff

- $F : \mathfrak{h}_n \to V$ is holomorphic,
- $F ((AZ + B)(CZ + D)^{-1}) = \rho(CZ + D)F(Z) \forall (\begin{pmatrix} A & B \\ C & D \end{pmatrix}) \in \Gamma^n$
- For $n = 1$, we need to assert a growth condition.
Let
\[ Q := \{ f = [a, b, c] = aX^2 + bXY + cY^2 \in \mathbb{Z}[X, Y] : f \geq 0 \}. \]

We have
\[
F(Z) = \sum_{f=[a,b,c] \in Q} C(f) e(\text{tr}(ZM_f))
\]
\[
= \sum_{f=[a,b,c] \in Q} C(f) e \left( \text{tr} \left( Z \left( \begin{array}{cc} a & b/2 \\ b/2 & c \end{array} \right) \right) \right)
\]
\[
= \sum_{f=[a,b,c] \in Q} C(f) e \left( a\tau + bz + c\tau' \right)
\]

where \( Z = \left( \begin{array}{cc} \tau & z \\ z & \tau' \end{array} \right) \) (\( \tau, \tau' \in \mathbb{H}_1 \) and \( z \in \mathbb{C} \)) and \( e(x) = e^{2\pi i x} \).

If the expansion of \( F \) is only supported on positive definite forms, it's a **cusp form**, i.e. \( F \in S_\rho^n \).
When $n = 1$, they are the same.

Siegel modular forms are multivariate modular forms.

For degree $> 1$, the growth condition comes for free (Koecher’s principle).

Fourier expansion supported on matrices.

For degree $> 1$ knowing coefficients allows computation of Hecke eigenvalues but not vice versa.

So really, unlike in the classical case, there are two different questions:

1. How do you compute Fourier coefficients?
2. How do you compute Hecke eigenvalues?
In this case $\rho = \det(\text{std})^k$ where $\text{std} : GL(2, \mathbb{C}) \to GL(\mathbb{C}^2)$ is inclusion and $k$ is some integer (since $\rho$ is rational).

Igusa: there are four generators for the ring of Siegel modular forms: an Eisenstein series $E_4$ of weight 4, $E_6$ of weight 6, and two cusp forms, $\chi_{10}, \chi_{12}$, one of weight 10 and one of weight 12.

Skoruppa: the four generators of even weight can be written down explicitly as Saito-Kurokawa lifts of $e_4, e_6, \Delta$. 
Igusa: a fifth generator $\chi_{35}$.

Ibukiyama: a formula for this generator

\[
\chi_{35} = \frac{1}{(2\pi i)^3} \begin{vmatrix}
4E_4 & 6E_6 & 10\chi_{10} & 12\chi_{12} \\
\frac{\partial E_4}{\partial \tau} & \frac{\partial E_6}{\partial \tau} & \frac{\partial \chi_{10}}{\partial \tau} & \frac{\partial \chi_{12}}{\partial \tau} \\
\frac{\partial E_4}{\partial \tau'} & \frac{\partial E_6}{\partial \tau'} & \frac{\partial \chi_{10}}{\partial \tau'} & \frac{\partial \chi_{12}}{\partial \tau'} \\
\frac{\partial z}{\partial \tau} & \frac{\partial z}{\partial \tau} & \frac{\partial \chi_{10}}{\partial \tau} & \frac{\partial \chi_{12}}{\partial \tau} \\
\frac{\partial z}{\partial \tau'} & \frac{\partial z}{\partial \tau'} & \frac{\partial \chi_{10}}{\partial \tau'} & \frac{\partial \chi_{12}}{\partial \tau'} \\
\end{vmatrix}.
\]
Sage Demo
Tsuyumine: A list of 34 generators for the ring of Siegel modular forms of level 1, degree 3 and scalar weight is known. It is not known if they are algebraically independent.

Poor-Yuen: Computations done in level 1, degree 4 and weights less than 16 via theta series. Everything so far appears to be a lift of some kind.
Several types of congruence subgroups. We restrict our attention to

$$\Gamma_0^{(2)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : C \equiv 0 \pmod{N} \right\}.$$
q be a positive definite quadratic form on a lattice \( L \) of rank \( n \).

Assume \( L \) is integral, i.e. \( q(L) \subseteq \mathbb{Z} \).

The theta series of degree 2 associated to \( L \) can be defined as

\[
\theta_L^{(2)}(Z) := \sum_{(v, w) \in L \times L} e\left(\text{tr}(ZM_{[v, w]})\right)
\]

where \( M_{[v, w]} := M_f \) for the binary quadratic form

\[
f = q(xv + yw).
\]

\( \theta_L^{(2)}(Z) \) is a Siegel modular form of degree 2 and weight \( n/2 \), for a certain level and character which can be easily determined.
Gathering coefficients, write

\[ \theta^{(2)}_L(Z) = \sum_f C(f) e(\text{tr}(ZM_f)). \]

With this notation,

\[ C(f) = \# \{(v, w) \in L \times L : f = q(xv + yw)\}. \]
Sage Demo
For some particular spaces, one can find lists of generators:

\[ \Gamma_0^{(2)}(2), \Gamma_0^{(2)}(3, \psi_3) \subset \Gamma_0^{(2)}(3), \Gamma_0^{(2)}(4, \psi_4) \subset \Gamma_0^{(2)}(4) \]

These lists consist of products of theta constants (these give the even weight generators) and forms determined via formulas like the one for \( \chi_{35} \).

For \( a, b \in \{0, 1\}^2 \), set

\[
\theta_{a,b}(Z) = \sum_{t \in \mathbb{Z}^2} e \left( \frac{1}{8} Z[2t + a] + \frac{1}{4} t(2t + a)b \right)
\]

where \( A[B] = ^t BAB \).
Sage Demo
Our Sage Package

- Co-written by Raum, Ryan, Skoruppa, Tornaría
- Original Cython and Sage Code
- Very general: can handle all the above as well as vector weight Siegel modular forms
Recall

\[ Q := \{ f = [a, b, c] = aX^2 + bXY + cY^2 \in \mathbb{Z}[X, Y] : f \geq 0 \}. \]

Note that \( f = [a, b, c] \) is positive semidefinite if and only if \( b^2 - 4ac \leq 0 \) and \( a, c \geq 0 \). We observe that \( Q \) is a semigroup with respect to usual addition. Moreover, GL has a natural left action on \( Q \):

\[ \text{GL} \times Q \rightarrow Q, \quad (A, f) \mapsto A.f := f((X, Y)A). \]

We make the following definition:

**Definition (Formal Siegel Modular Form)**

Let \( R \) be a module (or ring) with a GL left action. Then a Siegel modular form \( C \) is a map \( C : Q \rightarrow R \) for which \( C(A.f) = A.C(f) \) for all \( A \) in a finite index subgroup of GL.
In addition to everything already discussed,

- Saito-Kurokawa lifts: given a Jacobi form of weight $k$ and index 1
  
  $$
  \phi = \sum_{D, r \in \mathbb{Z}, D \leq 0 \atop D \equiv r^2 \pmod{4}} A_{\phi}(D) q^{(r^2-D)/4} \zeta^r
  $$

  we define

  $$
  C(f) = C([n, r, m]) := \begin{cases} 
  -\frac{B_{2k}}{4k} A_{\phi}(0) & f = 0 \\
  \sum a | (n, r, m) a^{k-1} A_{\phi} \left( \frac{r^2-4mn}{a^2} \right) & f = [n, r, m].
  \end{cases}
  $$

- Eisenstein as the S-K lift of the Jacobi Eisenstein series
- Jacobi $\vartheta$-function
<table>
<thead>
<tr>
<th>Group ( G )</th>
<th>Monoid ( M ) with ( G )-action</th>
<th>Module ( R ) with ( G )-action</th>
<th>Type of automorphic form</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \mathbb{Z}_{&gt;0} )</td>
<td>( \mathbb{F} )</td>
<td>elliptic modular forms</td>
</tr>
<tr>
<td>( \text{GL} )</td>
<td>( \mathbb{Q} )</td>
<td>( \mathbb{F}[X, Y]_{j}(\chi) )</td>
<td>vector-valued Siegel modular forms of degree 2</td>
</tr>
<tr>
<td>( \text{GL}(n, \mathbb{Z}) )</td>
<td>set of semi-positive definite integral quadratic forms ( f ) in ( n ) variables, ( (g, f) \mapsto f((X_1, \ldots, X_n)g) )</td>
<td>( \mathbb{F}[X_1, \ldots, X_n]_{j}, (g, p) \mapsto \operatorname{det}(g)^k \ p(((X_1, \ldots, X_n)g) )</td>
<td>vector-valued Siegel modular forms of degree ( n ) and weight ( k )</td>
</tr>
<tr>
<td>( \mathbb{Z}_L^* )</td>
<td>set of totally positive or zero elements in the inverse different of ( L ), ( (g, a) \mapsto g^2 a )</td>
<td>( \mathbb{F}, (g, r) \mapsto N(g)^k r )</td>
<td>Hilbert modular forms of (parallel) weight ( k ) over a totally real number field ( L )</td>
</tr>
<tr>
<td>( \text{GL}(n, \mathbb{Z}_L) )</td>
<td>set of semi-positive definite integral hermitian forms ( f ) over ( L ) with ( n ) variables, ( (g, f) \mapsto f((X_1, \ldots, X_n)g) )</td>
<td>( \mathbb{F}, (g, p) \mapsto \operatorname{det}(g)^k p )</td>
<td>Hermitian modular forms over the imaginary quadratic field ( L )</td>
</tr>
<tr>
<td>( \mathbb{Z}_L^* \times 2m\mathbb{Z}_L )</td>
<td>set of ( (D, r) ) in ( \vartriangle^{-2} \times \vartriangle^{-1} ) such that ( D \equiv r^2 \mod 4m\vartriangle^{-1} ), ( D = 0 ) or (-D \gg 0 ) (where ( \vartriangle ) denotes the different of ( L ), ( ((\varepsilon, x), (D, r)) \mapsto ((\varepsilon^2 D, \varepsilon(r + x)) )</td>
<td>( \mathbb{F}, ((\varepsilon, x), r) \mapsto N(\varepsilon)^k r )</td>
<td>Jacobi forms of weight ( k ) and index ( m \gg 0 ) (in ( \vartriangle^{-1} )) over a number field ( L )</td>
</tr>
</tbody>
</table>
If $F$ is a Hecke eigenform, then

$$\lambda_F(p)C([1, 1, 1]) = C([p, p, p]) + p^{k-2} \left( 1 + \left( \frac{p}{3} \right) \right)$$

and for

$$\Lambda_p := \lambda_F(p)^2 - \lambda_F(p)p^{k-2} \left( 1 + \left( \frac{p}{3} \right) \right) - p^{2k-3} + p^{2k-4} \left( \left( \frac{p}{3} \right) + \left( \frac{p}{3} \right) \right)^2$$

we have

$$\lambda_F(p^2)C([1, 1, 1]) = \Lambda_p C([1, 1, 1]) - p^{k-2} C([1, p, p]^2) - \sum_{\nu \bmod p} p^{k-2} C([1 + \nu + \nu^2, p(1 + 2\nu), p^2])$$

$$1 + \nu + \nu^2 \not\equiv 0 \bmod p$$
Computing the eigenvalue $\lambda_F(p)$ requires the computation of coefficients up to discriminant $3p^2$.

Computing the eigenvalue $\lambda_F(p^2)$ requires the computation of coefficients up to discriminant $O(p^4)$. 
For all even weights except 24 and 26, the characteristic polynomial of the Hecke operator $T_2$ is irreducible acting on the space of Siegel modular forms that are not Saito-Kurokawa lifts. This has been checked using this package for weights up to 88.
Sage Demo
Hecke eigenvalues, other methods

- Cunningham-Dembélé: Jacquet Langlands Correspondence to compute Hilbert-Siegel modular forms.
- Faber-van der Geer: compute trace of Hecke operators using analogue of Eichler Shimura
- Currently no modular symbol algorithm
Let $F$ be a nonzero Hecke eigenform. Let

$$L(F, s, \text{spin}) = \prod_{p \text{ prime}} L_p(F, p^{-s})^{-1}$$

where

$$L_p(F, X) = 1 - \lambda(p)X + (\lambda(p)^2 - \lambda(p^2) - p^{2k-4})X^2$$
$$- \lambda(p)p^{2k-3}X^3 + p^{4k-6}X^4$$

be the spinor $L$-function. $L(F, s, \text{spin})$ has an analytic continuation to the whole plane when $F$ is not a Saito-Kurokawa lift. When $F$ is a lift, it has a pole.
Let $\varepsilon(f)$ be the order the order of the unit group of the quadratic form $f$ and $A(D) := \left( \sum_{f > 0, \text{disc } f = D} C(f) / \varepsilon(f) \right)$. Then for negative fundamental discriminants $D$,

$$L(F, k - 1, \chi_D) = C_F |D|^{1-k} A(D)^2$$

where the LHS is the central critical value of the quadratic twist of $L(F, s)$.

Explicitly says that Fourier coefficients encode more data than eigenvalues.

Very recently checked using an optimized variant of our package for weights that have nonrational eigenforms.