Math 201

21 October 2008

Second Midterm

NAME (Print!): KEY

Check one: (1pm): ______

(2pm): ______

Problem	Points	Score
1	20	
2	20	
3	30	
4	20	
5	10	
Total	100	

Problem 1 (20 points): Newton's Law of Gravitation states that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GMm}{r^2}$$

where G is the gravitational constant and r is the distance between the two bodies.

(a) Find $\frac{dF}{dr}$ and explain its meaning. What does the minus sign indicate?

(b) Suppose that it is known Earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20,000$ km. How fast does this force change when $r = 10,000$ km.

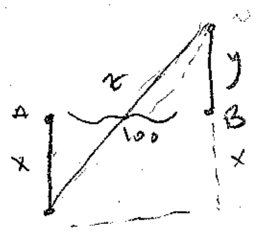
(a) $\frac{df}{dr} = \frac{-2\,GMm}{r^3}$

Means that F decreases as r increases

(b) $-2 = \frac{-2GMm}{(20000)^3} \Rightarrow GMm = +(20000)^3$

$\frac{dF}{dr} = \frac{-2\,(20000)^3}{(10000)^3} = -16$ N/km

Problem 2 (20 points): At noon ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and Ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 pm.



$$(x+y)^2+100^2=z^2$$

I know

$\frac{dx}{dt}$ $\frac{dy}{dt}$

$$2(x+y)\left(\frac{dx}{dt}+\frac{dy}{dt}\right)=2z\frac{dz}{dt}$$

I want $\frac{dz}{dt}$

After 4 hrs

$x = 140$

$y = 100$

$z = \sqrt{(240)^2+(100)^2} = 260$

So $2(240)(60) = 520\frac{dz}{dt}$

$$\frac{dz}{dt} = \frac{720}{13}$$

Problem 3 (30 points): Find $\frac{dy}{dx}$ for each of the following:

(a) $\tan(x-y) = \frac{y}{1+x^2}$

$$\frac{d}{dx}\tan(x-y) = \frac{d}{dx}\frac{y}{1+x^2}$$

$$\sec^2(x-y)\left(1-\frac{dy}{dx}\right) = \frac{\frac{dy}{dx}(1+x^2) - 2xy}{(1+x^2)^2}$$

$$\sec^2(x-y) - \sec^2(x-y)\frac{dy}{dx} = \frac{dy}{dx}\left(\frac{1+x^2}{(1+x^2)^2}\right) - \frac{2xy}{(1+x^2)^2}$$

$$\frac{\sec^2(x-y) + \frac{2xy}{(1+x^2)^2}}{\frac{1}{1+x^2} + \sec^2(x-y)} = \frac{dy}{dx}$$

(b) $y = 2^{3^{x^2}}$

$$y' = \ln 2 \; 2^{3^{x^2}} \left(3^{x^2}\right)' = \ln 2\left(2^{3^{x^2}}\right)\left(\ln 3 \; 3^{x^2}\right)\left(x^2\right)'$$

$$= \ln 2 \left(2^{3^{x^2}}\right) \ln 3 \left(3^{x^2}\right)(2x)$$

(c) $y = x^{e^x}$

$$\ln y = e^x \ln x$$

$$\frac{d}{dx}\ln y = \frac{d}{dx} e^x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = e^x \ln x + \frac{e^x}{x}$$

$$\frac{dy}{dx} = x^{e^x}\left(e^x \ln x + \frac{e^x}{x}\right)$$

Problem 4 (20 points): Prove the following differentiation rules:

(a) Using the limit definition of the derivative, prove $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.

$$\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$ since $\frac{1}{\sqrt{x+h}+\sqrt{x}}$ is continuous for all h.

(b) Show that for any real number n we have $\frac{d}{dx}x^n = nx^{n-1}$.

$$y = x^n$$

$$\ln y = n \cdot \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{x}$$

$$\frac{dy}{dx} = \frac{yn}{x} = n\frac{x^n}{x} = nx^{n-1}$$

Problem 5 (10 points): Find an equation of the tangent line to the curve $y = 4\sin^2(x)$ at the point $(\pi/6, 1)$.

$$\frac{dy}{dx} = 8\sin(x)\cos(x)$$

Slope at $\left(\frac{\pi}{6}, 1\right)$

$$\left.\frac{dy}{dx}\right|_{\frac{\pi}{6}} = 8\,\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = 8\cdot\frac{\sqrt{3}}{4} = 2\sqrt{3}$$

$$y-1 = 2\sqrt{3}\left(x-\frac{\pi}{6}\right).$$

(THIS PAGE INTENTIONALLY BLANK)