Lab 2

Statistical Uncertainties

Continuing Objectives

2. Know how to determine experimental uncertainties (multiple measurements of the same quantity, propagation of errors, etc.).

3. Be able to write an experimental result (including correct number of significant digits, uncertainty, units).

5. Know how to keep a clear and organized record, including an introduction (with purpose of lab and appropriate laws or equations), apparatus sketch, table of raw data and calculated quantities, and a good conclusion or summary.

7. Know how to make comparisons: are two measured quantities equal? Is a measured quantity statistically equivalent to a theoretical value?

Introduction

Successive measurements of any physical quantity always have some variation if read to enough significant figures. This variation can be caused by a number of factors, such as vibrations of mechanical parts, air currents, manufacturing variation between apparently identical components, variations in room temperature or humidity, electronic noise in instruments, and even random motions of atoms and molecules. An experimental result therefore always includes an estimate of these variations. In this lab, you will observe variations in repeated measurements. You will also learn how to report an experimental result accurately.
Part I: Statistics of M&Ms

Procedure (Part I)

You are to find an answer to the question: How many blue M&Ms are in a standard package? This number may vary from package to package. Your answer to the question must be expressed in such a way that someone could know whether or not their answer is statistically consistent with yours even though it might not be exactly the same. Note: this is exactly the kind of situation faced daily by people working in quality control labs in industry and by doctors trying to interpret changes in a patient’s lab results.

You will count the number of blue M&Ms in your bag. You will then share your results with others in class and find an appropriate way to express the class’ results.

1. Each student in the lab will be given a small bag of M&Ms. Open the bag over a clean piece of paper and count the number of blue M&Ms in your bag and record the number in the shared Google Sheet (link on the PHYS 211 website on the Lab Info page) as in Table 2.1.

2. Examine the table of values. Without doing any calculations, what is your estimate for the average number of blue M&Ms in a typical bag? How would you characterize the fluctuations in the values of your table? These fluctuations are a measure of the differences over the sample in the number of blue M&Ms in a single bag.

3. To interpret these data graphically, plot the number of blue M&Ms versus the student number making a scatter plot without lines connecting data points. Paste a copy of this plot into your lab notebook document. Does examination of this graph cause you to change your previous estimates made in step 2?

4. Next we answer the first question posed in step 2 quantitatively. As the best estimate for the number of blue M&Ms in a typical bag, calculate the average

<table>
<thead>
<tr>
<th>Student Number</th>
<th>Number of blue M&amp;Ms</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.1: Table to record class results for number of blue M&Ms in a bag.
Table 2.2: Data table for histogram. The sample data discussed in the manual is entered in the table on the right.

<table>
<thead>
<tr>
<th>Number of blue M&amp;Ms</th>
<th>Number of bags</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of blue M&amp;Ms</th>
<th>Number of bags</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>

value:

\[
\langle x \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i. \tag{2.1}
\]

Check that using the Excel function =AVERAGE(cell 1:cell N) will give you the same value of \( \langle x \rangle \) that you previously calculated. In future labs you may use this Excel function AVERAGE. Next draw your result for \( \langle x \rangle \) as a horizontal line on your plot from step 3. You can do this on your document by drawing a line (found under Insert>Shapes on Microsoft Word) over your graph at the appropriate place. The average \( \langle x \rangle \) is also called the experimental mean.

5. Your plot from step 3 is one way to look at the complete set of values. Note that your plot includes more information than just the experimental mean; the plot also includes the fluctuations. There is another way of graphically representing your data called a histogram. Make a second table in your Excel spreadsheet as shown in Table 2.2.

The first column is the number of blue M&Ms in a bag. The second column tells you how many bags had that specific number of blue M&Ms. For example, if no student found a bag with 0 blue M&Ms, 3 students found bags with 8 blue M&Ms, and 4 students found bags with 9 blue M&Ms, then your table would have the entries shown on the right in Table 2.2.(NOTE: You can use the Excel function =COUNTIF(cell 1:cell N, condition) to do this. For instance to determine the number of students finding 4 blue M&Ms, enter =COUNTIF(A1:A30, 4).)

Complete the entries in your Excel version of Table 2.2 which corresponds to the M&M data obtained in step 1. Check that your numbers in the second column add up to the total number of students. Now plot your data from your
version of Table 2.2 (number of bags versus number of blue M&Ms). This plot is the \textit{histogram} of your data.

Paste a copy of this histogram plot into your lab notebook. Draw in a vertical line at the \( \langle x \rangle \) value you obtained previously.

In step 3 we asked you to characterize the fluctuation in the values of Table 1. How would you graphically represent these fluctuations on your histogram?

6. Now we are ready to address these fluctuations quantitatively. We want a quantity which tells us how much the \( x_i \) values differ from \( \langle x \rangle \), so let’s consider \( (x_i - \langle x \rangle) \). However, we do not care about the sign, so let’s take the square of this quantity, \( (x_i - \langle x \rangle)^2 \). It would then make sense to take the average of these values by summing all \( N \) values of the quantity above and dividing by \( N \), where \( N \) is the total number of independent pieces of information that are used in the calculation. However, as we saw in Eq. 2.1, the calculation of \( \langle x \rangle \) depends on \( x_i \). Because of this, there are now \( N - 1 \) independent measurements in the calculation; for example, \( x_1 \) could be determined from \( \langle x \rangle \) and \( x_2, x_3, x_4 \), and so on. Therefore, we instead divide by \( N - 1 \) (the term \( N - 1 \) is called Bessel’s correction). Finally, we must take the square root of this expression. We call this resulting quantity the experimental standard deviation:

\[
\text{experimental standard deviation} = s = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \langle x \rangle)^2}. \tag{2.2}
\]

7. Add to your Table 2.1 a third column that calculates values of \( (x_i - \langle x \rangle)^2 \). Using the values in this third column, determine the experimental standard deviation, \( s \), using Eq. (2.2).

8. Check that using the Excel function \texttt{=STDEV(cell1:cellN)} gives you the same value of \( s \) that you previously calculated from your spreadsheet. That means that in future labs you may use the Excel function \texttt{STDEV}. Compare this new value of \( s \) with your previous estimates of fluctuations in steps 3 and 5. Remember that these fluctuations are a measure of the uncertainty in the number of blue M&Ms in a single bag.

9. You are now ready to answer the question: “If you were to open one single new M&M bag, what would be your prediction for the number of blue M&Ms in this bag?” Report your result as explained in the Appendix A under the section \textit{Reporting a Numerical Result}.
10. You can transform your histogram into a probability distribution of blue M&Ms. To do so, add to your Table 2.2 another column which is

\[ P(x) = \frac{\text{number of bags with } x \text{ blue M&Ms}}{\text{total number of bags}}. \]  

(2.3)

Plot a graph of \( P(x) \) versus the number of blue M&Ms, \( x \). Describe in words the meaning of \( P(x) \).

Theory

The universe of possible measurements

So far we have only opened one M&M bag for each student (\( N \) bags of M&Ms in total). In order to fully characterize the complete distribution of blue M&Ms in this batch, our goal would be to open every single M&M bag produced in this batch. Besides getting a stomach pain from eating all these M&Ms, this task is obviously impossible. Even though that task is impossible, we won’t give up. Instead, we imagine how the probability distribution \( P(x) \) of the blue M&Ms of all (let’s say \( N \to \infty \)) M&M bags would look.

We call this distribution the distribution of the universe of possible measurements. Just as before it would have a mean value and a standard deviation, the true mean value \( \mu \) is:

\[ \mu = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i; \]  

(2.4)

and the true standard deviation \( \sigma \) is:

\[ \sigma = \lim_{N \to \infty} \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2}; \]  

(2.5)

as illustrated in Figure 2.1.
When the distribution has the typical Gaussian shape of Figure 2.1, approximately 68% of the measurements lie within $\pm \sigma$ of the true mean, 95% within $\pm 2\sigma$ of the true mean, and 99.74% within $\pm 3\sigma$ of the true mean.

Your job as an experimenter is to estimate the true mean and the true standard deviation from just a few measurements. If you make just one measurement, that measurement has a 68% chance of lying within $\pm \sigma$ of the true mean. But this statement tells you nothing because neither the true mean nor the true standard deviation is known ahead of time. In fact, a single measurement, while giving a ballpark estimate of the true mean, says nothing at all about the true standard deviation. It is for this reason that, almost always, you must make multiple measurements.

Estimates of the true mean and true standard deviation

The best estimates of $\mu$ and $\sigma$ that can be obtained from a set of $N$ measurements are given by the experimental mean and standard deviation of your $N$ measurements. The experimental mean and standard deviation are exactly what you have determined so far in this lab, i.e., as given in Eq. (2.1) and Eq. (2.2).
Standard deviation of the mean

While $s$ tells you how close an individual measurement is likely to be to the true mean, we would like to know how far the experimental mean $\langle x \rangle$ is from the true mean. Because $\langle x \rangle$ is an average of $N$ measurements, and is calculated from measurements that lie both above and below the true mean, you might correctly guess that $\langle x \rangle$ is likely to lie closer to the true mean than a typical individual measurement. But how much closer? For large values of $N$ (greater than approximately 10), it turns out that the true means has a 68% chance of lying within $\pm s/\sqrt{N}$ of $\langle x \rangle$. Here $N$, as in Eq. (7.1), is the number of values used to obtain the mean.

The uncertainty of the mean, given by $s/\sqrt{N}$, is called the standard deviation of the mean. This is an accurate name, for $s/\sqrt{N}$ is the best estimate of what you'd get if you measured the mean of $N$ measurements many times and then computed the experimental standard deviation of these means.

So, whenever you quote the uncertainty of a quantity that you’ve measured $N$ times, you should quote the standard deviation of the mean,

$$\text{standard deviation of the mean} = \frac{s}{\sqrt{N}}. \quad (2.6)$$

11. Determine the standard deviation of the mean in this experiment.

12. In step 9, you were asked to answer the question, “If you were to open one single new M&M bag, what is your prediction for the number of blue M&Ms in this bag?” Now we ask you to answer a different question: “If everybody in the class were given a new bag of M&Ms, what is your prediction for the class average of the number of blue M&Ms per bag?” Report your answer in the correct format as described in Appendix A.

STOP

Show your estimate to your instructor or TA before continuing.

13. Write a mini-conclusion for Part I. You should include an explanation of the difference between standard deviation and standard deviation of the mean.
Part II: Does a Pendulum’s Period Depend on Amplitude?

You are provided with a pendulum and a stopwatch accurate to the nearest hundredth of a second. You are to determine whether the period of oscillation of the pendulum (the time for one complete swing) is different for small amplitude swings than it is for large amplitude swings.

Method

Call $T_L$ the measured time for the pendulum to swing through 10 complete swings with large amplitude (about 15°). Similarly, call $T_S$ the measured time for 10 swings of small amplitude (about 5°). The numerical value, $T_L - T_S$, represents one measurement of the difference between the two times. When you measure $T_L$ and $T_S$ repeatedly you should expect to get many different values of $T_L - T_S$. What is the correct result for this difference? Is it different from zero? If so, is the difference real or just due to chance? These are some of the questions that can be answered by a statistical analysis of your data, as in Part I.

Procedure

In timing the swings of the pendulum, be as precise as possible. For example, it’s better to start timing after the pendulum has swung a couple of times so that timing the start will be exactly like timing the finish. And don’t make the mistake of having measuring all the values of $T_L$ in a row and then all the values for $T_S$ in a row. This could introduce a bias due to practice with timing one particular situation repeatedly and getting “better” at it. Finally, consider whether it’s better to start and stop at the top of a swing, where the pendulum is momentarily at rest, or at the bottom, where it’s traveling fastest.

When you have thought about these considerations, come up with a plan for your method of measurement and describe it to your instructor or lab assistant. Record your method in your notebook.

1. Use the marks on the wall as starting locations for large swings (about 15° from vertical) and small swings (about 5° from vertical). Then take one measurement of the time $T_L$ for ten large swings and one measurement of the time $T_S$ for ten small swings. Calculate the difference $T_L - T_S$. (You use the time for ten swings rather than just one to reduce the relative effect of uncertainties in starting and stopping the stopwatch.) Continue to alternate measurements
of $T_L$ and $T_S$ until you have 12 values of $T_L - T_S$. (Alternation is important because any improvement due to practice will tend to apply equally to $T_L$ and $T_S$.)

2. Since you are interested in whether the true mean of $T_L - T_S$ is different from zero, calculate the experimental mean of $T_L - T_S$ and the standard deviation of the mean. For these calculations, use an Excel spreadsheet as in Part I.

3. Write your result for $T_L - T_S$ in standard form (see Appendix A).

4. Based on this result, conclude whether $T_L$ is or is not different from $T_S$, and hence, whether the period of a pendulum depends on the size of its amplitude. Write this conclusion and your reasoning in your lab notebook document and discuss it with your lab instructor or TA.