J.R.Taylor, "Classical Mechanics", University Science Books, Sausalito 2005.

5.4 Damped Oscillations

I shall now return to the one-dimensional oscillator, and take up the possibility that there are resistive forces that will damp the oscillations. There are several possibilities for the resistive force. Ordinary sliding friction is approximately constant in magnitude, but always directed opposite to the velocity. The resistance offered by a fluid, such as air or water, depends on the velocity in a complicated way. However, as we saw in Chapter 2, it is sometimes a reasonable approximation to assume that the resistive force is proportional to v or (under different circumstances) to v^2. Here I shall assume that the resistive force is proportional to v; specifically, $\mathbf{f} = -b\mathbf{v}$. One of my main reasons is that this case leads to an especially simple equation to solve, and the equation is itself a very important equation that appears in several other contexts and is therefore well worth studying.[6]

Consider, then, an object in one dimension, such as a cart attached to a spring, that is subject to a Hooke's law force, $-kx$, and a resistive force, $-b\dot{x}$. The net force on the object is $-b\dot{x} - kx$, and Newton's second law reads (if I move the two force terms over to the left side)

$$m\ddot{x} + b\dot{x} + kx = 0. \tag{5.24}$$

One of the beautiful things about physics is the way the same mathematical equation can arise in totally different physical contexts, so that our understanding of the equation in one situation carries over immediately to the other. Before we set about solving Equation (5.24), I would like to show how the same equation appears in the study of LRC circuits. An LRC circuit is a circuit containing an inductor (inductance L), a capacitor (capacitance C), and a resistor (resistance R), as sketched in Figure 5.10. I have chosen the positive direction for the current to be counterclockwise, and the charge $q(t)$ to be the charge on the left-hand plate of the capacitor [with $-q(t)$ on the right], so that $I(t) = \dot{q}(t)$. If we follow around the circuit in the positive direction, the electric potential drops by $L\dot{I} = L\ddot{q}$ across the inductor, by $RI = R\dot{q}$ across the resistor, and by q/C across the capacitor. Applying Kirchoff's second rule for circuits, we conclude that

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0. \tag{5.25}$$

This has exactly the form of Equation (5.24) for the damped oscillator, and anything that we learn about the equation for the oscillator will be immediately applicable to the LRC circuit. Notice that the inductance L of the electric circuit plays the role of the mass of the oscillator, the resistance term $R\dot{q}$ corresponds to the resistive force, and $1/C$ to the spring constant k.

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[6] You should be aware, however, that although the case I am considering — that the resistive force $\mathbf{f}$ is linear in $\mathbf{v}$ — is very important, it is nevertheless a *very* special case. I shall describe some of the startling complications that can occur when $\mathbf{f}$ is not linear in $\mathbf{v}$ in Chapter 12 on nonlinear mechanics and chaos.

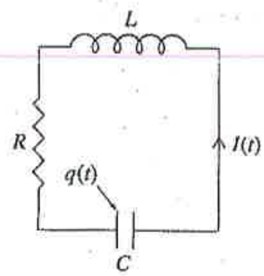

Figure 5.10 An LRC circuit.

Let us now return to mechanics and the differential equation (5.24). To solve this equation it is convenient to divide by m and then introduce two other constants. I shall rename the constant b/m as 2β,

$$\frac{b}{m} = 2\beta. \tag{5.26}$$

This parameter β, which can be called the **damping constant**, is simply a convenient way to characterize the strength of the damping force — as with b, large β corresponds to a large damping force and conversely. I shall rename the constant k/m as ω_o^2, that is,

$$\omega_o = \sqrt{\frac{k}{m}}. \tag{5.27}$$

Notice that ω_o is precisely what I was calling ω in the previous two sections. I have added the subscript because, once we admit resistive forces, various other frequencies become important. From now on, I shall use the notation ω_o to denote the system's **natural frequency**, the *frequency at which it would oscillate if there were no resistive force present*, as given by (5.27). With these notations, the equation of motion (5.24) for the damped oscillator becomes

$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0$

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0. \tag{5.28}$$

Notice that both of the parameters β and ω_o have the dimensions of inverse time, that is, frequency.

Equation (5.28) is another second-order, linear, homogeneous equation [the last was (5.4)]. Therefore, if by any means we can spot two independent[7] solutions, $x_1(t)$ and $x_2(t)$ say, then any solution must have the form $C_1x_1(t) + C_2x_2(t)$. What this

[7] It is about time I gave you a definition of "independent." In general this is a little complicated, but for two functions it is easy: Two functions are independent if neither is a constant multiple of the other. Thus the two functions $\sin(x)$ and $\cos(x)$ are independent; likewise the two functions x and x^2; but the two functions x and $3x$ are not.

means is that we are free to play a game of inspired guessing to find ourselves two independent solutions; if by hook or by crook we can spot two solutions, then we have the general solution.

In particular, there is nothing to stop us *trying* to find a solution of the form

$$x(t) = e^{rt} \tag{5.29}$$

for which

$$\dot{x} = re^{rt}$$

and

$$\ddot{x} = r^2 e^{rt}.$$

Substituting into (5.28) we see that our guess (5.29) satisfies (5.28) if and only if

$(r+\beta)^2 - \beta^2 + \omega_0^2 = 0$

$$r^2 + 2\beta r + \omega_0^2 = 0 \tag{5.30}$$

[an equation sometimes called the **auxiliary equation** for the differential equation (5.28)]. The solutions of this equation are, of course, $r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$. Thus if we define the two constants

$$\begin{aligned} r_1 &= -\beta + \sqrt{\beta^2 - \omega_0^2} \\ r_2 &= -\beta - \sqrt{\beta^2 - \omega_0^2} \end{aligned} \tag{5.31}$$

then the two functions $e^{r_1 t}$ and $e^{r_2 t}$ *are* two independent solutions of (5.28) and the general solution is

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{5.32}$$

$$= e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2 - \omega_0^2}\,t} + C_2 e^{-\sqrt{\beta^2 - \omega_0^2}\,t}\right). \tag{5.33}$$

This solution is rather too messy to be especially illuminating, but, by examining various ranges of the damping constant β, we can begin to see what (5.33) entails.

Undamped Oscillation

If there is no damping then the damping constant β is zero, the square root in the exponents of (5.33) is just $i\omega_0$, and our solution reduces to

$$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}, \tag{5.34}$$

the familiar solution for the undamped harmonic oscillator.

Weak Damping

Suppose next that the damping constant β is small. Specifically, suppose that

$$\beta < \omega_0, \tag{5.35}$$

a condition sometimes called **underdamping**. In this case, the square root in the exponents of (5.33) is again imaginary, and we can write

$$\sqrt{\beta^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \beta^2} = i\omega_1,$$

where

$$\boxed{\omega_1 = \sqrt{\omega_0^2 - \beta^2}.} \tag{5.36}$$

The parameter ω_1 is a frequency, which is less than the natural frequency ω_o. In the important case of very weak damping ($\beta \ll \omega_o$), ω_1 is very close to ω_o. With this notation, the solution (5.33) becomes

$$x(t) = e^{-\beta t}\left(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right). \tag{5.37}$$

This solution is the product of two factors: The first, $e^{-\beta t}$, is a decaying exponential, which steadily decreases toward zero. The second factor has exactly the form (5.34) of undamped oscillations, except that the natural frequency ω_o is replaced by the somewhat lower frequency ω_1. We can rewrite the second factor, as in Equation (5.11), in the form $A\cos(\omega_1 t - \delta)$ and our solution becomes

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta). \tag{5.38}$$

This solution clearly describes simple harmonic motion of frequency ω_1 with an exponentially decreasing amplitude $Ae^{-\beta t}$, as shown in Figure 5.11. The result (5.38) suggests another interpretation of the damping constant β. Since β has the dimensions of inverse time, $1/\beta$ is a time, and we now see that it is the time in which the amplitude function $Ae^{-\beta t}$ falls to $1/e$ of its initial value. Thus, at least for underdamped oscillations, β can be seen as the decay parameter, a measure of the rate at which the motion dies out,

$$\text{(decay parameter)} = \beta \qquad \text{[underdamped motion]}.$$

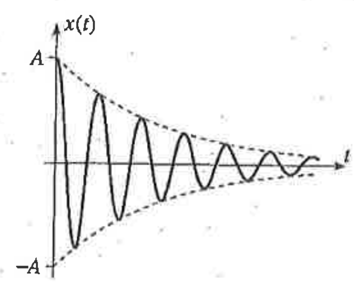

Figure 5.11 Underdamped oscillations can be thought of as simple harmonic oscillations with an exponentially decreasing amplitude $Ae^{-\beta t}$. The dashed curves are the envelopes, $\pm Ae^{-\beta t}$.

The larger β the more rapidly the oscillations die out, at least for the case $\beta < \omega_o$ that we are discussing here.

Strong Damping

Suppose instead that the damping constant β is large. Specifically suppose that

$$\beta > \omega_o, \tag{5.39}$$

a condition sometimes called **overdamping**. In this case, the square root in the exponents of (5.33) is real and our solution is

$$x(t) = C_1 e^{-\left(\beta - \sqrt{\beta^2-\omega_o^2}\right)t} + C_2 e^{-\left(\beta + \sqrt{\beta^2-\omega_o^2}\right)t}. \tag{5.40}$$

Here we have two real exponential functions, both of which decrease as time goes by (since the coefficients of t in both exponents are negative). In this case, the motion is so damped that it completes no bona fide oscillations. Figure 5.12 shows a typical case in which the oscillator was given a kick from O at $t = 0$; it slid out to a maximum displacement and then slid ever more slowly back again, returning to the origin only in the limit that $t \to \infty$. The first term on the right of (5.40) decreases more slowly than the second, since the coefficient in its exponent is the smaller of the two. Thus the long-term motion is dominated by this first term. In particular, the rate at which the motion dies out can be characterized by the coefficient in the first exponent,

$$(\text{decay parameter}) = \beta - \sqrt{\beta^2 - \omega_o^2} \qquad [\text{overdamped motion}]. \tag{5.41}$$

Careful inspection of (5.41) shows that — contrary to what one might expect — the rate of decay of overdamped motion gets smaller if the damping constant β is made bigger. (See Problem 5.20.)

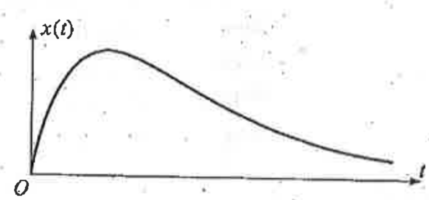

Figure 5.12 Overdamped motion in which the oscillator is kicked from the origin at $t = 0$. It moves out to a maximum displacement and then moves back toward O asymptotically as $t \to \infty$.

Critical Damping

The boundary between underdamping and overdamping is called **critical damping** and occurs when the damping constant is equal to the natural frequency, $\beta = \omega_o$. This case has some interesting features, especially from a mathematical point of view. When $\beta = \omega_o$ the two solutions that we found in (5.33) are the same solution, namely

$$x(t) = e^{-\beta t}. \tag{5.42}$$

[This happened because the two solutions of the auxiliary equation (5.30) happen to coincide when $\beta = \omega_o$.] This is the one case where our inspired guess, to seek a solution of the form $x(t) = e^{rt}$, fails to find us two solutions of the equation of motion, and we have to find a second solution by some other method. Fortunately, in this case, it is not hard to spot a second solution: As you can easily check, the function

$$x(t) = te^{-\beta t} \tag{5.43}$$

is also a solution of the equation of motion (5.28) in the special case that $\beta = \omega_o$. (See Problems 5.21 and 5.24.) Thus the general solution for the case of critical damping is

$\dot{x}(t) = e^{-\beta t} + (-\beta)t e^{-\beta t}$ $\ddot{x}(t) = -\beta e^{-\beta t} - \beta e^{-\beta t} + \beta^2 t e^{-\beta t}$

$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = e^{-\beta t}(-\beta - \beta + \beta^2 t + 2\beta - 2\beta^2 t + \beta^2 t) = 0$

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}. \tag{5.44}$$

Notice that both terms contain the same exponential factor $e^{-\beta t}$. Since this factor is what dominates the decay of the oscillations as $t \to \infty$, we can say that both terms decay at about the same rate, with decay parameter

$$(\text{decay parameter}) = \beta = \omega_o \qquad [\text{critical damping}].$$

It is interesting to compare the rates at which the various types of damped oscillation die out. We have seen that in each case, this rate is determined by a "decay parameter," which is just the coefficient of t in the exponent of the dominant exponential factor in $x(t)$. Our findings can be summarized as follows:

damping	β	decay parameter
none	$\beta = 0$	0
under	$\beta < \omega_o$	β
critical	$\beta = \omega_o$	β
over	$\beta > \omega_o$	$\beta - \sqrt{\beta^2 - \omega_o^2}$

Figure 5.13 is a plot of the decay parameter as a function of β and shows clearly that the motion dies out most quickly when $\beta = \omega_o$; that is, when the damping is critical. There are situations where one wants any oscillations to die out as quickly as possible. For example, one wants the needle of an analog meter (a voltmeter or pressure gauge, for instance) to settle down rapidly on the correct reading. Similarly, in a car, one wants the oscillations caused by a bumpy road to decay quickly. In such cases one must arrange for the oscillations to be damped (by the shock absorbers in a car), and for the quickest results the damping should be reasonably close to critical.

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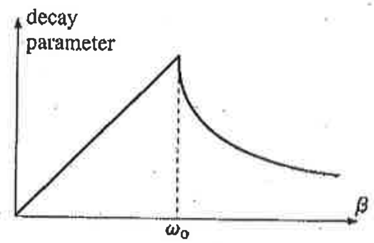

Figure 5.13 The decay parameter for damped oscillations as a function of the damping constant β. The decay parameter is biggest, and the motion dies out most quickly, for critical damping, with $\beta = \omega_o$.

5.5 Driven Damped Oscillations

Any natural oscillator, left to itself, eventually comes to rest, as the inevitable damping forces drain its energy. Thus if one wants the oscillations to continue, one must arrange for some external "driving" force to maintain them. For example, the motion of the pendulum in a grandfather clock is driven by periodic pushes caused by the clock's weights; the motion of a young child on a swing is maintained by periodic pushes from a parent. If we denote the external driving force by $F(t)$ and if we assume as before that the damping force has the form $-bv$, then the net force on the oscillator is $-bv - kx + F(t)$ and the equation of motion can be written as

$$m\ddot{x} + b\dot{x} + kx = F(t). \tag{5.45}$$

Like its counterpart for undriven oscillations, this differential equation crops up in several other areas of physics. A prominent example is the LRC circuit of Figure 5.10. If we want the oscillating current in that circuit to persist, we must apply a driving EMF, $\mathcal{E}(t)$, in which case the equation of motion for the circuit becomes

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = \mathcal{E}(t) \tag{5.46}$$

in perfect correspondence with (5.45).

As before, we can tidy Equation (5.45) if we divide the equation by m and replace b/m by 2β and k/m by ω_o^2. In addition, I shall denote $F(t)/m$ by

$$f(t) = \frac{F(t)}{m}, \tag{5.47}$$

the force per unit mass. With this notation, (5.45) becomes

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f(t) \tag{5.48}$$