From Optics, E. Hecht, 3rd ed (Addison-Wesley, 1998)

7 The Superposition of Waves

In succeeding chapters we shall study the phenomena of polarization, interference, and diffraction. These all share a common conceptual basis in that they deal, for the most part, with various aspects of the same process. Stating this in the simplest terms, we are really concerned with what happens when two or more lightwaves overlap in some region of space. The precise circumstances governing this superposition determine the final optical disturbance. Among other things we are interested in learning how the specific properties of each constituent wave (amplitude, phase, frequency, etc.) influence the ultimate form of the composite disturbance.

Recall that each field component of an electromagnetic wave (E_x, E_y, E_z, B_x, B_y, and B_z) satisfies the scalar three-dimensional differential wave equation,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \qquad [2.60]$$

A significant feature of this expression is that it is *linear*; $\psi(\mathbf{r}, t)$ and its derivatives appear only to the first power. Consequently, if $\psi_1(\mathbf{r}, t)$, $\psi_2(\mathbf{r}, t)$, ... , $\psi_n(\mathbf{r}, t)$ are individual solutions of Eq. (2.59), *any linear combination* of them will, in turn, be a solution. Thus

$$\psi(\mathbf{r}, t) = \sum_{i=1}^{n} C_i \psi_i(\mathbf{r}, t) \qquad (7.1)$$

satisfies the wave equation, where the coefficients C_i are simply arbitrary constants. Known as the ***Principle of Superposition***, this property suggests that the resultant disturbance at any point in a medium is the algebraic sum of the separate constituent waves (Fig. 7.1). At this time we are interested

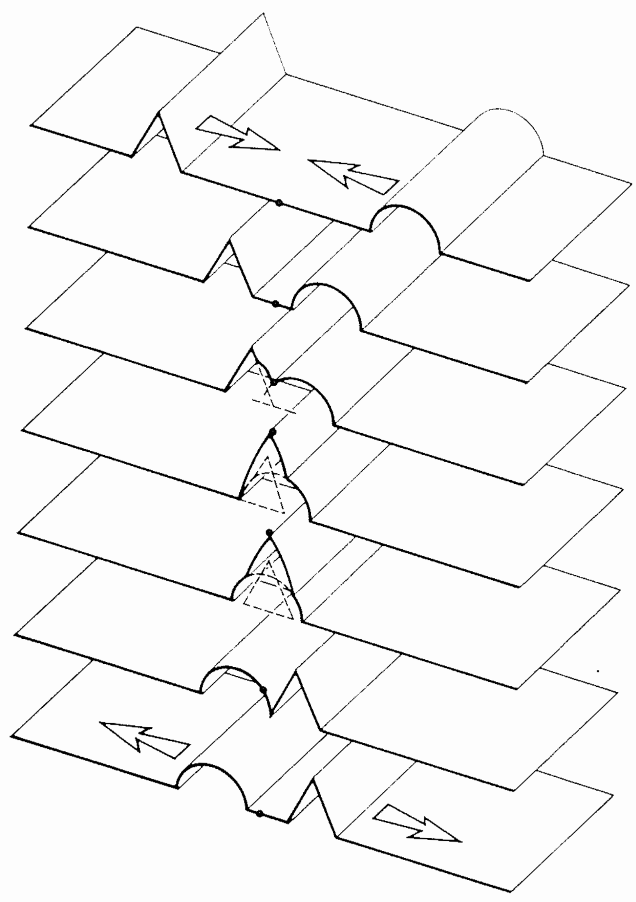

FIGURE 7.1 The superposition of two disturbances.

only in linear systems where the superposition principle is applicable. Do keep in mind, however, that large-amplitude waves, whether sound waves or waves on a string, can generate a nonlinear response. The focused beam of a high-intensity laser (where the electric field might be as high as 10^{10} V/cm) is easily capable of eliciting nonlinear effects (see Chapter 13). By comparison, the electric field associated with sunlight here on Earth has an amplitude of only about 10 V/cm.

In many instances we need not be concerned with the vector nature of light, and for the present we will restrict ourselves to such cases. For example, if the lightwaves all propagate along the same line and share a common constant plane of vibration, they can each be described in terms of one electric-field component. These would all be either parallel or antiparallel at any instant and could thus be treated as scalars. A good deal more will be said about this point as we progress; for now, let's represent the optical disturbance as a scalar function $E(\mathbf{r}, t)$, which is a solution of the differential wave equation. This approach leads to a simple scalar theory that is highly useful as long as we are careful about applying it.

7.1 The Addition of Waves of the Same Frequency

There are several equivalent ways of mathematically adding two or more overlapping waves that have the same frequency and wavelength. Let's examine these different approaches so that, in any particular situation, we can use the one most suitable.

7.1.1 The Algebraic Method

A solution of the differential wave equation can be written in the form

$$E(x, t) = E_0 \sin [\omega t - (kx + \varepsilon)] \tag{7.2}$$

in which E_0 is the amplitude of the harmonic disturbance propagating along the positive x-axis. To separate the space and time parts of the phase, let

$$\alpha(x, \varepsilon) = -(kx + \varepsilon) \tag{7.3}$$

so that

$$E(x, t) = E_0 \sin [\omega t + \alpha(x, \varepsilon)] \tag{7.4}$$

Suppose then that there are two such waves

$$E_1 = E_{01} \sin (\omega t + \alpha_1) \tag{7.5a}$$

and

$$E_2 = E_{02} \sin (\omega t + \alpha_2) \tag{7.5b}$$

each with the same frequency and speed, coexisting in space. The resultant disturbance is the linear superposition of these waves:

$$E = E_1 + E_2$$

or, on expanding Eqs. (7.5a) and (7.5b)

$$E = E_{01} (\sin \omega t \cos \alpha_1 + \cos \omega t \sin \alpha_1) + E_{02} (\sin \omega t \cos \alpha_2 + \cos \omega t \sin \alpha_2)$$

When we separate out the time-dependent terms, this becomes

$$E = (E_{01} \cos \alpha_1 + E_{02} \cos \alpha_2) \sin \omega t + (E_{01} \sin \alpha_1 + E_{02} \sin \alpha_2) \cos \omega t \tag{7.6}$$

Since the parenthetical quantities are constant in time, let

$$E_0 \cos \alpha = E_{01} \cos \alpha_1 + E_{02} \cos \alpha_2 \tag{7.7}$$

and

$$E_0 \sin \alpha = E_{01} \sin \alpha_1 + E_{02} \sin \alpha_2 \tag{7.8}$$

This is not an obvious substitution, but it will be legitimate as long as we can solve for E_0 and α. To that end, square and add Eqs. (7.7) and (7.8) to get

$$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02} \cos (\alpha_2 - \alpha_1) \tag{7.9}$$

and divide Eq. (7.8) by (7.7) to get

$$\tan \alpha = \frac{E_{01} \sin \alpha_1 + E_{02} \sin \alpha_2}{E_{01} \cos \alpha_1 + E_{02} \cos \alpha_2} \tag{7.10}$$

Provided these last two expressions are satisfied for E_0 and α, the situation of Eqs. (7.7) and (7.8) is valid. The total disturbance [Eq. (7.6)] then becomes

$$E = E_0 \cos \alpha \sin \omega t + E_0 \sin \alpha \cos \omega t$$

or

$$E = E_0 \sin (\omega t + \alpha) \tag{7.11}$$

A single disturbance results from the superposition of the sinusoidal waves E_1 and E_2. *The composite wave [Eq. (7.11)] is harmonic and of the same frequency as the constituents, although its amplitude and phase are different.*

Note that when $E_{01} >> E_{02}$ in Eq. (7.10), $\alpha \approx \alpha_1$ and when $E_{02} >> E_{01}$, $\alpha \approx \alpha_2$; the resultant is in-phase with the dominant component wave (take another look at Fig. 4.11). The flux density of a lightwave is proportional to its amplitude squared, by way of Eq. (3.44). It follows from Eq. (7.9) that the resultant flux density is not simply the sum of the component flux densities; there is an additional contribution $2E_{01}E_{02} \cos (\alpha_2 - \alpha_1)$, known as the **interference term**. The crucial factor is the difference in phase between the two *interfering* waves E_1 and E_2, $\delta \equiv (\alpha_2 - \alpha_1)$. When $\delta = 0, \pm 2\pi, \pm 4\pi, \ldots$ the resultant amplitude is a maximum, whereas $\delta = \pm\pi, \pm 3\pi, \ldots$ yields a minimum (Problem 7.3). In the former case, the waves are said to be in-phase; crest overlaps crest. In the latter instance, the waves are 180° out-of-phase and trough overlaps crest, as shown in Fig. 7.2. Realize that the *phase difference* may arise from a difference in path length traversed by the two waves, as well as a difference in the initial phase angle; that is,

$$\delta = (kx_1 + \varepsilon_1) - (kx_2 + \varepsilon_2) \tag{7.12}$$

or

$$\delta = \frac{2\pi}{\lambda} (x_1 - x_2) + (\varepsilon_1 - \varepsilon_2) \tag{7.13}$$

Here x_1 and x_2 are the distances from the sources of the two waves to the point of observation, and λ is the wavelength in the pervading medium. If the waves are initially in-phase at their respective emitters, then $\varepsilon_1 = \varepsilon_2$, and

$$\delta = \frac{2\pi}{\lambda} (x_1 - x_2) \tag{7.14}$$

This would also apply to the case in which two disturbances from the same source traveled different routes before arriving at the point of observation. Since $n = c/v = \lambda_0/\lambda$,

$$\delta = \frac{2\pi}{\lambda_0} n(x_1 - x_2) \tag{7.15}$$

The quantity $n(x_1 - x_2)$ is known as the **optical path difference** and will be represented by the abbreviation OPD or by the symbol Λ. It's the difference in the two optical path lengths [Eq. (4.9)]. It is possible, in more complicated situations, for each wave to travel through a number of different thicknesses of different media (Problem 7.6). Notice too that $\Lambda/\lambda_0 = (x_1 - x_2)/\lambda$ is the number of waves in the medium corresponding to the path difference; one route is that many wavelengths longer than the other. Since each wavelength is

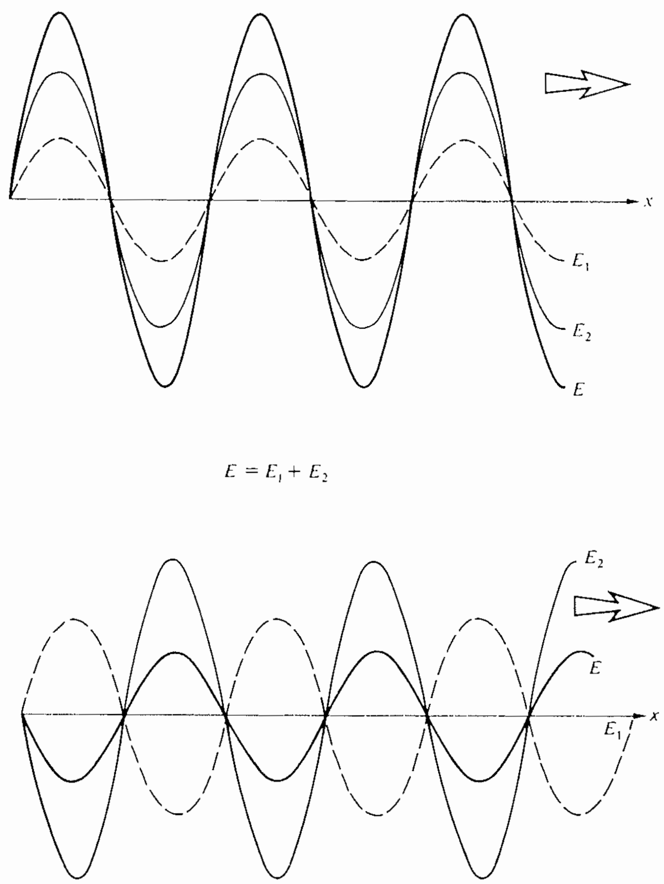

FIGURE 7.2 The superposition of two harmonic waves in-phase and out-of-phase.

associated with a 2π radian phase change, $\delta = 2\pi(x_1 - x_2)/\lambda$, or

$$\delta = k_0\Lambda \tag{7.16}$$

k_0 being the propagation number in vacuum; that is, $2\pi/\lambda_0$. One route is essentially δ radians longer than the other.

Waves for which $\varepsilon_1 - \varepsilon_2$ is *constant*, regardless of its value, are said to be **coherent**, a situation we shall assume obtains throughout most of this discussion.

One special case of some interest is the superposition of the waves

$$E_1 = E_{01} \sin [\omega t - k(x + \Delta x)]$$

and

$$E_2 = E_{02} \sin (\omega t - kx)$$

where in particular $E_{01} = E_{02}$ and $\alpha_2 - \alpha_1 = k\,\Delta x$. It is left to Problem 7.7 to show that in this case Eqs. (7.9), (7.10), and (7.11) lead to a resultant wave of

$$E = 2E_{01} \cos \left(\frac{k\,\Delta x}{2}\right) \sin \left[\omega t - k\left(x + \frac{\Delta x}{2}\right)\right] \tag{7.17}$$

This brings out rather clearly the dominant role played by the path length difference, Δx, especially when the waves are emitted in-phase ($\varepsilon_1 = \varepsilon_2$). There are many practical instances in which one arranges just these conditions, as will be seen later. If $\Delta x << \lambda$, the resultant has an amplitude that is nearly $2E_{01}$, whereas if $\Delta x = \lambda/2$, it is zero. Recall that the former situation (p. 21) is referred to as **constructive interference**, and the latter as **destructive interference** (see Fig. 7.3).

By repeated applications of the procedure used to arrive at Eq. (7.11), we can show that the ***superposition of any number of coherent harmonic waves having a given frequency and traveling in the same direction leads to a harmonic wave of that same frequency*** (Fig. 7.4). We happen to have chosen to represent the two waves above in terms of sine functions, but the same results would prevail if cosine functions were used. In general, then, the sum of N such waves,

$$E = \sum_{i=1}^{n} E_{0i} \cos (\alpha_i \pm \omega t)$$

is given by

$$E = E_0 \cos (\alpha \pm \omega t) \tag{7.18}$$

where

$$E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2 \sum_{j>i}^{N} \sum_{i=1}^{N} E_{0i}E_{0j} \cos (\alpha_i - \alpha_j) \tag{7.19}$$

and

$$\tan \alpha = \frac{\sum_{i=1}^{N} E_{0i} \sin \alpha_i}{\sum_{i=1}^{N} E_{0i} \cos \alpha_i} \tag{7.20}$$

Pause for a moment and satisfy yourself that these relations are indeed true.

Consider a number (N) of atomic emitters constituting an ordinary source (an incandescent bulb, candle flame, or discharge lamp). A flood of light is emitted that presumably corresponds to a torrent of photons, which manifest themselves *en masse* as an electromagnetic wave. To keep things in a wave perspective, it's useful to imagine the photon as some-

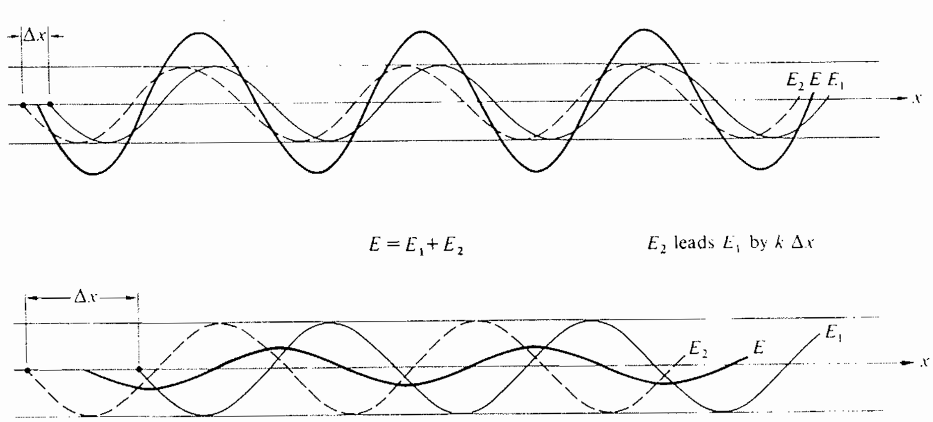

FIGURE 7.3 Waves out-of-phase by $k\Delta x$ radians.

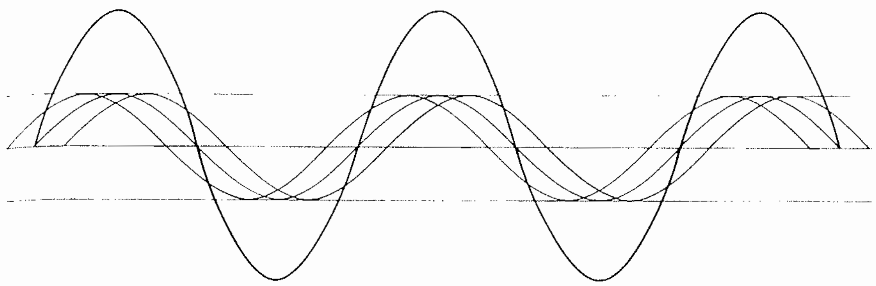

FIGURE 7.4 The superposition of three harmonic waves yields a harmonic wave of the same frequency.

how associated with a short-duration oscillatory wave pulse. Each atom is effectively an independent source of photon wavetrains (Section 3.4.4), and these, in turn, extend in time for roughly 1 to 10 ns. In other words, the atoms can be thought of as emitting wavetrains that have a sustained phase for only up to about 10 ns. After that a new wavetrain may be emitted with a totally random phase, and it too will be sustained for less than approximately 10 ns, and so forth. On the whole each atom emits a disturbance (composed of a stream of photons) that varies in its phase rapidly and randomly.

In any event, the phase of the light from one atom, $\alpha_i(t)$, will remain constant with respect to the phase from another atom $\alpha_j(t)$, for only a time of at most 10 ns before it changes randomly: the atoms are coherent for up to about 10^{-8} s. Since flux density is proportional to the time average of E_0^2, generally taken over a comparatively long interval of time, it follows that the second summation in Eq. (7.19) will contribute terms proportional to $\langle\cos[\alpha_i(t) - \alpha_j(t)]\rangle$, each of which will average out to zero because of the random rapid nature of the phase changes. Only the first summation remains in the time average, and its terms are constants. If each atom is emitting wavetrains of the same amplitude E_{01}, then

$$E_0^2 = NE_{01}^2 \tag{7.21}$$

The resultant flux density arising from N sources having random, rapidly varying phases is given by N times the flux density of any one source. In other words, *it is determined by the sum of the individual flux densities.*

A flashlight bulb, whose atoms are all emitting a random tumult, puts out light which (as the superposition of these essentially "incoherent" wavetrains) is itself rapidly and randomly varying in phase. Thus two or more such bulbs will emit light that is essentially incoherent (i.e., for durations longer than about 10 ns), light whose total combined irradiance will simply equal the sum of the irradiances contributed by each individual bulb. This is also true for candle flames, flashbulbs, and all thermal (as distinct from laser) sources. ***We cannot expect to see interference when the lightwaves from two reading lamps overlap.***

At the other extreme, if the sources are coherent and in-phase at the point of observation (i.e., $\alpha_i = \alpha_j$), Eq. (7.19) will become

$$E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2\sum_{j>i}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}$$

or, equivalently,

$$E_0^2 = \left(\sum_{i=1}^{N} E_{0i}\right)^2 \tag{7.22}$$

Again, supposing that each amplitude is E_{01}, we get

$$E_0^2 = (NE_{01})^2 = N^2E_{01}^2 \tag{7.23}$$

In this case of in-phase coherent sources, we have a situation in which the amplitudes are added first and then squared to determine the resulting flux density. The superposition of coherent waves generally has the effect of altering the spatial distribution of the energy but not the total amount present. If there are regions where the flux density is greater than the sum of the individual flux densities, there will be regions where it is less than that sum.

7.1.2 The Complex Method

It is often mathematically convenient to make use of the complex representation when dealing with the superposition of harmonic disturbances. The wave

$$E_1 = E_{01}\cos(kx \pm \omega t + \varepsilon_1)$$

or

$$E_1 = E_{01}\cos(\alpha_1 \mp \omega t)$$

can then be written as

$$\tilde{E}_1 = E_{01}e^{i(\alpha_1 \mp \omega t)} \tag{7.24}$$

if we remember that we are interested only in the real part (see Section 2.5). Suppose that there are N such overlapping waves having the same frequency and traveling in the *positive x-direction*. The resultant wave is given by

$$\tilde{E} = E_0 e^{i(\alpha + \omega t)}$$

which is equivalent to Eq. (7.18) or, upon summation of the component waves,

$$\tilde{E} = \left[\sum_{j=1}^{N} E_{0j}e^{i\alpha_j}\right]e^{+i\omega t} \tag{7.25}$$

The quantity

$$E_0 e^{i\alpha} = \sum_{j=1}^{N} E_{0j}e^{i\alpha_j} \tag{7.26}$$

is known as the *complex amplitude* of the composite wave and is simply the sum of the complex amplitudes of the constituents. Since

$$E_0^2 = (E_0 e^{i\alpha})(E_0 e^{i\alpha})^* \tag{7.27}$$

we can always compute the resultant irradiance from Eqs. (7.26) and (7.27). For example, if $N = 2$,

$$E_0^2 = (E_{01}e^{i\alpha_1} + E_{02}e^{i\alpha_2})(E_{01}e^{-i\alpha_1} + E_{02}e^{-i\alpha_2})$$

whence

$$E_0^2 = E_{01}^2 + E_{02}^2 + E_{01}E_{02}[e^{i(\alpha_1 - \alpha_2)} + e^{-i(\alpha_1 - \alpha_2)}]$$

or

$$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_1 - \alpha_2)$$

which is identical to Eq. (7.9).

7.1.3 Phasor Addition

The summation described in Eq. (7.26) can be represented graphically as an addition of vectors in the complex plane (recall the discussion on p. 23). In the parlance of electrical engineering, the complex amplitude is known as a **phasor**, and it's specified by its magnitude and phase, often written simply as $E_0\angle\alpha$. Imagine, then, that we have a disturbance described by

$$E_1 = E_{01}\sin(\omega t + \alpha_1)$$

In Fig. 7.5*a* the wave is represented by a vector of length E_{01} rotating counterclockwise at a rate ω such that its projection on the vertical axis is $E_{01}\sin(\omega t + \alpha_1)$. If we were concerned with cosine waves, we would take the projection on the horizontal axis. Incidentally, the rotating vector is, of course, a phasor $E_{01}\angle\alpha_1$, and the R and I designations signify the real and imaginary axes. Similarly, a second wave

$$E_2 = E_{02}\sin(\omega t + \alpha_2)$$

is depicted along with E_1 in Fig. 7.5*b*. Their algebraic sum, $E = E_1 + E_2$, is the projection on the I-axis of the resultant

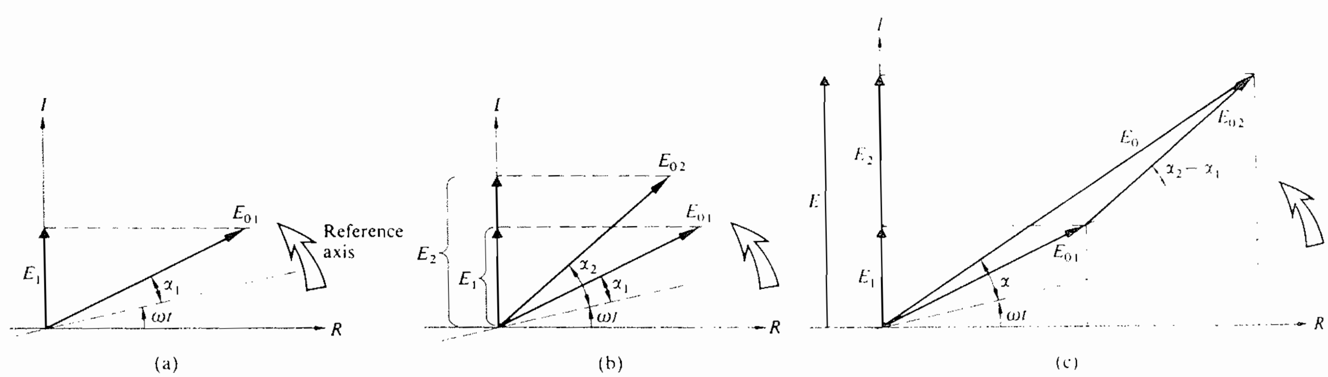

FIGURE 7.5 Phasor addition.

phasor determined by the vector addition of the component phasors, as in Fig. 7.5*c*. The law of cosines applied to the triangle of sides E_{01}, E_{02}, and E_0 yields

$$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_2 - \alpha_1)$$

where use was made of the fact that $\cos[\pi - (\alpha_2 - \alpha_1)] = -\cos(\alpha_2 - \alpha_1)$. This is identical to Eq. (7.9), as it must be. Using the same diagram, observe that $\tan\alpha$ is given by Eq. (7.10) as well. We are usually concerned with finding E_0 rather than $E(t)$, and since E_0 is unaffected by the constant revolving of all the phasors, it will often be convenient to set $t = 0$ and eliminate that rotation.

Some rather elegant schemes, such as the *vibration curve* and the *Cornu spiral* (Chapter 10), will be predicated on the technique of phasor addition. As a final example, let's briefly examine the wave resulting from the addition of

$$E_1 = 5\sin\omega t$$

$$E_2 = 10\sin(\omega t + 45°)$$

$$E_3 = \sin(\omega t - 15°)$$

$$E_4 = 10\sin(\omega t + 120°)$$

and

$$E_5 = 8\sin(\omega t + 180°)$$

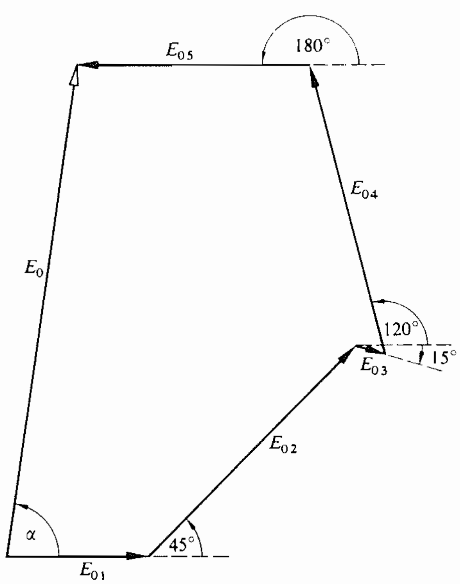

FIGURE 7.6 The sum of E_1, E_2, E_3, E_4, and E_5.

where ω is in degrees per second. The appropriate phasors $5\angle 0°$, $10\angle 45°$, $1\angle -15°$, $10\angle 120°$, and $8\angle 180°$ are plotted in Fig. 7.6. Notice that each phase angle, whether positive or negative, is referenced to the horizontal. One need only read off $E_0\angle\alpha$ with a scale and protractor to get $E = E_0\sin(\omega t + \alpha)$. It is evident that this technique offers a tremendous advantage in speed and simplicity, if not in accuracy.

7.1.4 Standing Waves

We saw earlier (p. 21) that the sum of solutions to the differential wave equation is itself a solution. Thus, in general,

$$\psi(x, t) = C_1 f(x - vt) + C_2 g(x + vt)$$

satisfies the differential wave equation. In particular let's examine *two harmonic waves of the same frequency propagating in opposite directions.* A situation of practical concern arises when the incident wave is reflected backward off some sort of mirror; a rigid wall will do for sound waves or a conducting sheet for electromagnetic waves. Imagine that an incoming wave traveling to the left,

$$E_I = E_{0I}\sin(kx + \omega t + \varepsilon_I) \tag{7.28}$$

strikes a mirror at $x = 0$ and is reflected to the right in the form

$$E_R = E_{0R}\sin(kx - \omega t + \varepsilon_R) \tag{7.29}$$

The composite wave in the region to the right of the mirror is $E = E_I + E_R$. In other words, the two waves (one traveling to the right, the other to the left) exist simultaneously in the region between the source and the mirror.

We could perform the indicated summation and arrive at a general solution* much like that of Section 7.1. However, some valuable physical insights can be gained by taking a slightly more restricted approach.

The initial phase ε_I may be set to zero by merely starting our clock at a time when $E_I = E_{0I}\sin kx$. Certain qualifications determined by the physical setup must be met by the mathematical solution, and these are known formally as ***boundary conditions***. For example, if we were talking about a

*See, for example, J. M. Pearson, *A Theory of Waves*.

rope with one end tied to a wall at $x = 0$, that point must always have a zero displacement. The two overlapping waves, one incident and the other reflected, would have to add in such a way as to yield a zero resultant wave at $x = 0$. Similarly, at the boundary of a perfectly conducting sheet the resultant electromagnetic wave must have a zero electric-field component parallel to the surface. Assuming $E_{0I} = E_{0R}$, the boundary conditions require that at $x = 0$, $E = 0$, and since $\varepsilon_I = 0$, it follows from Eqs. (7.28) and (7.29) that $\varepsilon_R = 0$. The composite disturbance is then

$$E = E_{0I} [\sin (kx + \omega t) + \sin (kx - \omega t)]$$

Applying the identity

$$\sin \alpha + \sin \beta = 2 \sin \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta)$$

yields

$$E(x, t) = 2E_{0I} \sin kx \cos \omega t \qquad (7.30)$$

This is the equation for a **standing** or **stationary wave**, as opposed to a traveling wave (Fig. 7.7). Its profile does not move through space; it is clearly not of the form $f(x \pm vt)$. At any point $x = x'$, the amplitude is a constant equal to $2E_{0I} \sin kx'$, and $E(x', t)$ varies harmonically as $\cos \omega t$. At certain points, namely, $x = 0, \lambda/2, \lambda, 3\lambda/2, \ldots$, the disturbance will be zero at all times. These are known as **nodes** or *nodal points* (Fig. 7.8). Halfway between each adjacent node, that is, at $x = \lambda/4, 3\lambda/4, 5\lambda/4, \ldots$, the amplitude has a maximum value of $\pm 2E_{0I}$, and these points are known as the **antinodes**. The disturbance $E(x, t)$ will be zero at all values of x whenever $\cos \omega t = 0$, that is, when $t = (2m + 1)\tau/4$, where $m = 0, 1, 2, 3, \ldots$ and τ is the period of the component waves.

If the reflection off the mirror is not perfect, as is often the case, the composite wave will contain a traveling component along with the stationary wave. Under such conditions there will be a net transfer of energy, whereas for the pure standing wave there is none.

Although the analysis carried out above is essentially one-dimensional, standing waves exist in two and three dimensions as well. The phenomenon is extremely commonplace: standing waves occur in one dimension on guitar strings and diving boards, in two dimensions on the surface of a drum or in a jiggled pail of water (Fig. 7.9), and in three dimensions when you sing in a shower stall. In fact, standing waves are created within the cavities inside your head whenever you sing, no matter where you are.

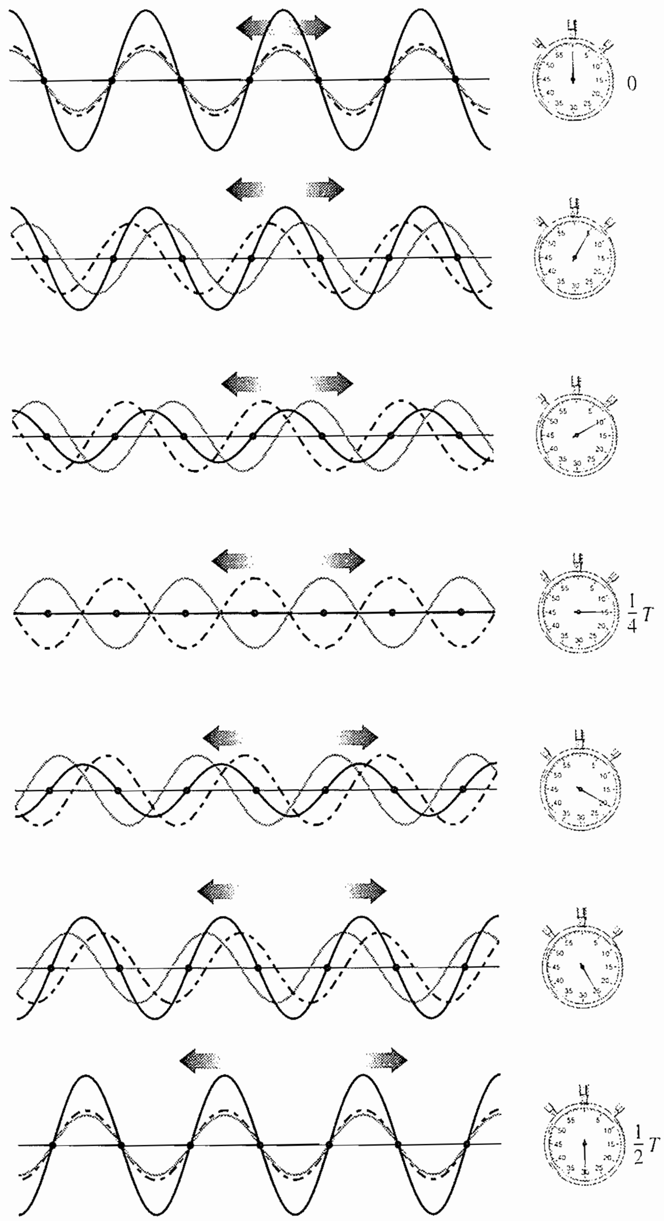

FIGURE 7.7 The creation of standing waves. Two waves of the same amplitude and wavelength traveling in opposite directions form a stationary disturbance that oscillates in place.

If a standing-wave system is driven by an oscillating source, it will efficiently absorb energy provided that the vibrations match one of its standing-wave modes. That process is known as resonance, and it happens every time your

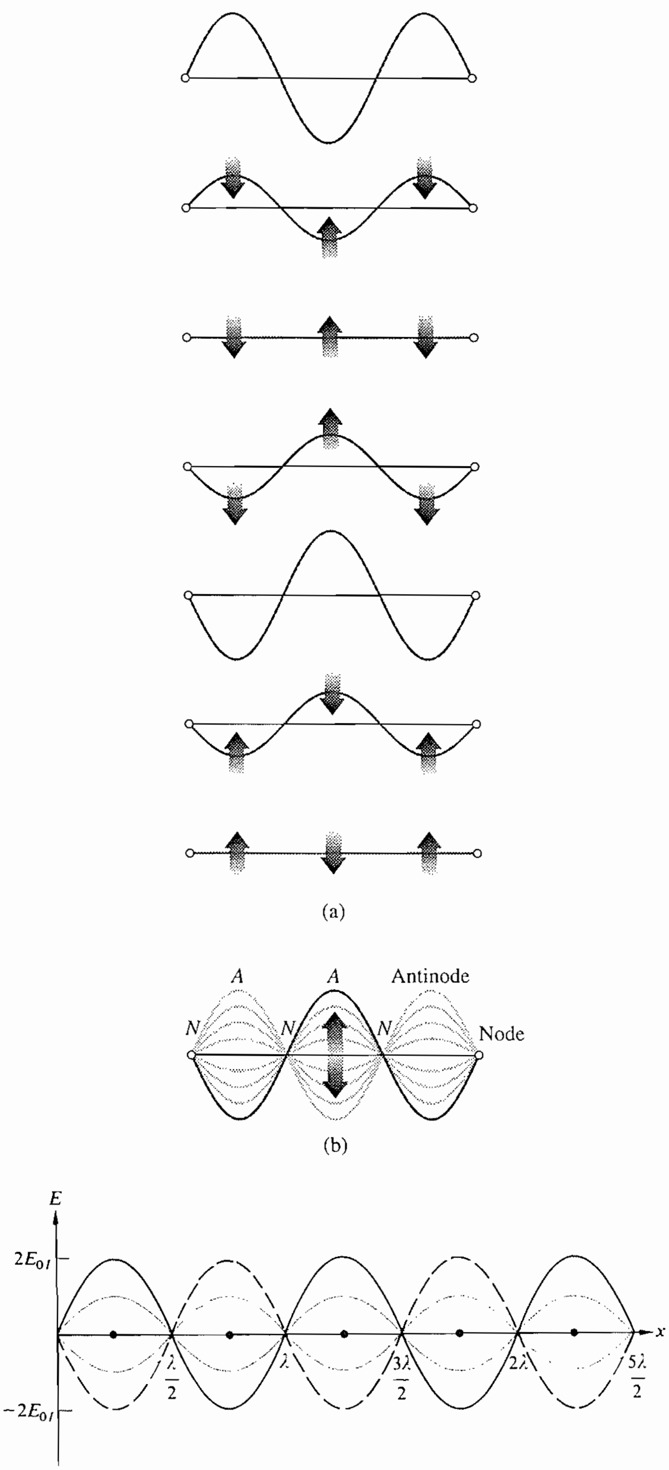

FIGURE 7.8 A standing wave at various times.

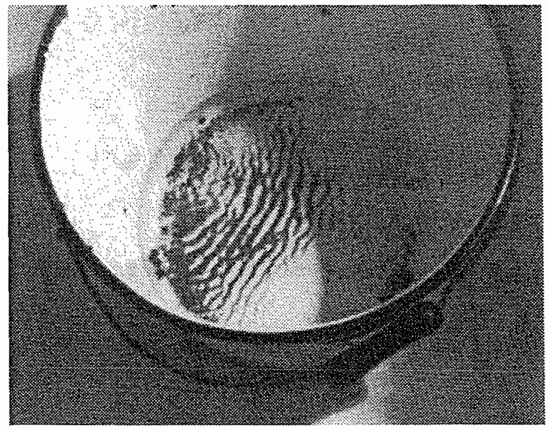

FIGURE 7.9 A pail used to wash a floor contained a suspension of fine dirt particles in water. When placed in a curved sink, the pail gently rocked along a fixed axis, setting up standing waves and distributing the particles in ridges as they settled.

house buzzes when an airplane flies low overhead or when a heavy truck passes by. If the source continues to supply energy, the wave will continue to build until the system's inherent losses equal the energy input and equilibrium is reached. This ability to sustain and simplify an input is an extremely important feature of standing-wave systems. The ear's auditory canal is just such a resonant cavity. It amplifies (by about 100%) sounds in the range from $\approx$3 kHz to $\approx$4 kHz. Similarly, the laser builds its powerful emission within a standing-wave cavity (p. 585). Figure 7.10 shows the standing-wave pattern produced when a reflecting rod is placed in front of an antenna emitting $\approx$3 GHz electromagnetic waves.

It was by measuring the distances between the nodes of standing waves that Hertz was able to determine the wavelength of the radiation in his historic experiments (see Section 3.6.1). A few years later, in 1890, Otto Wiener first demonstrated the existence of standing lightwaves. The arrangement he used is depicted in Fig. 7.11. It shows a normally incident parallel beam of quasimonochromatic light reflecting off a front-silvered mirror. A transparent photographic film, less than $\lambda/20$ thick, deposited on a glass plate, was inclined to the mirror at an angle of about 10^{-3} radians. In that way the film plate cut across the pattern of standing plane waves. After developing the emulsion, it was found to be blackened along a series of equidistant parallel bands. These corresponded to the regions where the photographic layer had intersected the antinodal planes. Significantly, there was no blackening of the emulsion at the mirror's surface. It can be shown that the

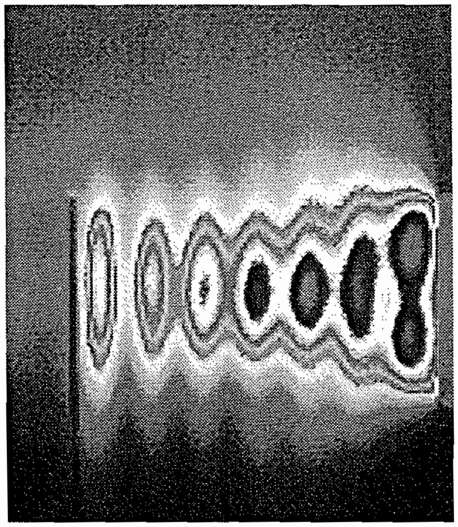

FIGURE 7.10 A two-dimensional standing-wave pattern formed between a source and a reflector. EM-waves from a 3.9-GHz antenna enter from the right. They reflect off a metal rod and travel back to the antenna. The pattern is made visible by absorbing the microwave radiation and recording the resulting temperature distribution with an IR camera. (Photo courtesy H. H. Pohle, Phillips Laboratory, Kirtland Air Force Base.)

nodes and antinodes of the magnetic field component of an electromagnetic standing wave alternate with those of the electric field (Problem 7.10). We might suspect as much from the fact that at $t = (2m + 1)\tau/4$, $E = 0$ for all values of x, so to conserve energy it follows that $B \neq 0$. In agreement with theory, Hertz had previously (1888) determined the existence of a nodal point of the electric field at the surface of his reflector. Accordingly, Wiener could conclude that the blackened regions were associated with antinodes of the **E**-field. ***It is the electric field that triggers the photochemical process.***

In a similar way Drude and Nernst showed that the **E**-field is responsible for fluorescence. These observations are all quite understandable, since the force exerted on an electron by the **B**-field component of an electromagnetic wave is generally negligible in comparison to that of the **E**-field. It is for these reasons that the electric field is referred to as the *optic disturbance* or *light field*.

Standing waves generated by two oppositely propagating disturbances represent a special case of the broader subject of double-beam interference (p. 377). Consider the two point sources sending out waves in Fig. 7.12. When point P, the point of observation somewhere near the middle is far from the sources, angle ϕ is small, the two waves superimpose, and there results a complicated interference pattern (that will be treated in detail in Chapter 9). Suffice it to say here that the space surrounding the sources will be filled with a system of bright and dark bands where the interference is alternately constructive and destructive. As P comes closer and ϕ gets larger, the fringes become finer, that is, narrower, until P is on

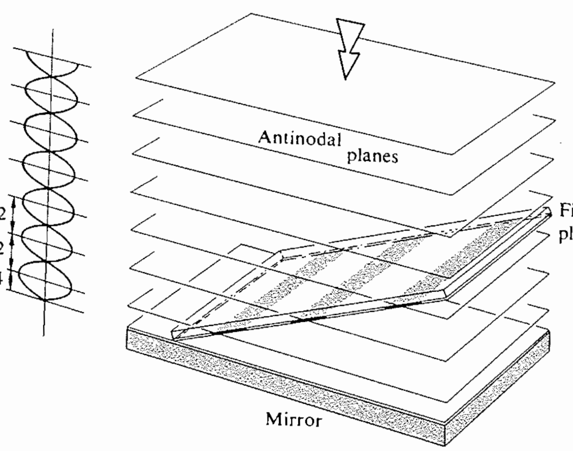

FIGURE 7.11 Wiener's experiment.

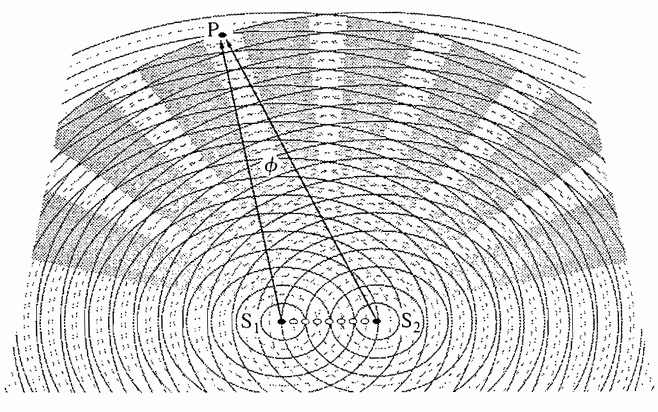

FIGURE 7.12 Two monochromatic point sources. At any point P the resultant wave is maximum where peak (—) overlaps peak (—) or trough (– –) overlaps trough (– –). It's minimum where peak overlaps trough. The maxima that form along the $\overline{S_1S_2}$ line correspond to standing waves.

the line joining the sources and $\phi = 180°$. Then standing waves are set up, and the "fringes" are the finest they'll get, namely, half a wavelength peak-to-peak.

7.2 The Addition of Waves of Different Frequency

Thus far the analysis has been restricted to the superposition of waves all having the same frequency. Yet one never actually has disturbances, of any kind, that are strictly monochromatic. It will be far more realistic, as we shall see, to speak of quasi-monochromatic light, which is composed of a narrow range of frequencies. The study of such light will lead us to the important concepts of bandwidth and coherence time.

The ability to modulate light effectively (Section 8.11.3) makes it possible to couple electronic and optical systems in a way that has had and will certainly continue to have far-reaching effects on the entire technology. Moreover, with the advent of electro-optical techniques, light has taken on a significant role as a carrier of information. This section is devoted to developing some of the mathematical ideas needed to appreciate this new emphasis.

7.2.1 Beats

Consider the composite disturbance arising from a combination of the waves

$$E_1 = E_{01} \cos (k_1 x - \omega_1 t)$$

and

$$E_2 = E_{01} \cos (k_2 x - \omega_2 t)$$

which have equal amplitudes and zero initial phase angles. The net wave

$$E = E_{01}[\cos (k_1 x - \omega_1 t) + \cos (k_2 x - \omega_2 t)]$$

can be reformulated as

$$E = 2E_{01} \cos \tfrac{1}{2}[(k_1 + k_2)x - (\omega_1 + \omega_2)t] \times \cos \tfrac{1}{2}[(k_1 - k_2)x - (\omega_1 - \omega_2)t]$$

using the identity

$$\cos \alpha + \cos \beta = 2 \cos \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta)$$

Now define the quantities $\bar{\omega}$ and $\bar{k}$, which are the ***average angular frequency*** and ***average propagation number***, respectively. Similarly, the quantities ω_m and k_m are designated the ***modulation frequency*** and ***modulation propagation number***, respectively. Let

$$\bar{\omega} \equiv \tfrac{1}{2}(\omega_1 + \omega_2) \qquad \omega_m \equiv \tfrac{1}{2}(\omega_1 - \omega_2) \qquad (7.31)$$

and

$$\bar{k} \equiv \tfrac{1}{2}(k_1 + k_2) \qquad k_m \equiv \tfrac{1}{2}(k_1 - k_2) \qquad (7.32)$$

thus

$$E = 2E_{01} \cos (k_m x - \omega_m t) \cos (\bar{k}x - \bar{\omega}t) \qquad (7.33)$$

The total disturbance may be regarded as a traveling wave of frequency $\bar{\omega}$ having a time-varying or modulated amplitude $E_0(x, t)$ such that

$$E(x, t) = E_0(x, t) \cos (\bar{k}x - \bar{\omega}t) \qquad (7.34)$$

where

$$E_0(x, t) = 2E_{01} \cos (k_m x - \omega_m t) \qquad (7.35)$$

In applications of interest here, ω_1 and ω_2 will always be rather large. In addition, if they are comparable to each other, $\omega_1 \approx \omega_2$, then $\bar{\omega} >> \omega_m$ and $E_0(x, t)$ will change slowly, whereas $E(x, t)$ will vary quite rapidly (Fig. 7.13). The irradiance is proportional to

$$E_0^2(x, t) = 4E_{01}^2 \cos^2 (k_m x - \omega_m t)$$

or

$$E_0^2(x, t) = 2E_{01}^2[1 + \cos (2k_m x - 2\omega_m t)]$$

Notice that $E_0^2(x, t)$ oscillates about a value of $2E_{01}^2$ with an angular frequency of $2\omega_m$ or simply $(\omega_1 - \omega_2)$, which is known as the **beat frequency**. That is, E_0 varies at the modulation frequency, whereas E_0^2 varies at twice that, namely, the beat frequency.

Beats were first observed with the use of light in 1955 by Forrester, Gudmundsen, and Johnson.* To obtain two waves

*A. T. Forrester, R. A. Gudmundsen, and P. O. Johnson, "Photo-electric Mixing of Incoherent Light." *Phys. Rev.* **99**, 1691 (1955).

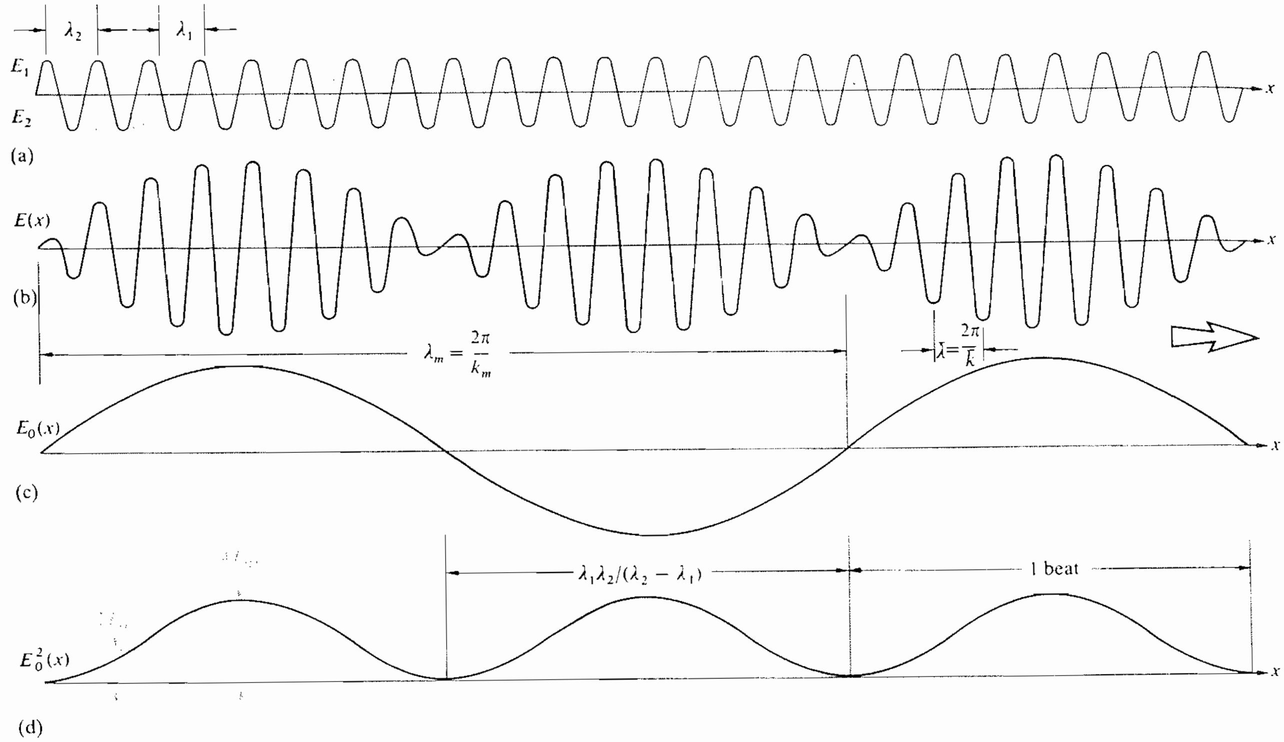

FIGURE 7.13 The superposition of two harmonic waves of different frequency.

of slightly different frequency they used the Zeeman Effect. When the atoms of a discharge lamp, in this case mercury, are subjected to a magnetic field, their energy levels split. As a result, the emitted light contains two frequency components, ν_1 and ν_2, which differ in proportion to the magnitude of the applied field. When these components are recombined at the surface of a photoelectric mixing tube, the beat frequency, $\nu_1 - \nu_2$, is generated. Specifically, the field was adjusted so that $\nu_1 - \nu_2 = 10^{10}$ Hz, which conveniently corresponds to a 3-cm microwave signal. The recorded photoelectric current had the same form as the $E_0^2(x)$ curve in Fig. 7.13*d*.

The advent of the laser has since made the observation of beats using light considerably easier. Even a beat frequency of a few Hz out of 10^{14} Hz can be seen as a variation in phototube current. The observation of beats now represents a particularly sensitive and fairly simple means of detecting small frequency differences. The ring laser (Section 9.8.3), functioning as a gyroscope, utilizes beats to measure frequency differences induced as a result of the rotation of the system. The Doppler Effect, which accounts for the frequency shift when light is reflected off a moving surface, provides another series of applications of beats. By scattering light off a target, whether solid, liquid, or even gaseous, and then beating the original and reflected waves, we get a precise measure of the target speed. In much the same way on an atomic scale, laser light will shift in phase upon interacting with sound waves moving in a material. (This phenomenon is called Brillouin Scattering.) Thus $2\omega_m$ becomes a measure of the speed of sound in the medium.

7.2.2 Group Velocity

The specific relationship between ω and k determines v, the phase velocity of a wave. In a nondispersive medium such as vacuum [Eq. (2.33)] $v = \omega/k$ and **a plot of ω versus k is a**