The Binomial Distribution

In Chapter 3 of *Measurements and their Uncertainties*, Hughes and Hase discuss both the Gaussian probability distribution function and the Poisson probability function. The Poisson distribution is actually a special-case limit of a more general probability distribution, the binomial distribution, which is the subject of these notes.

The binomial distribution is applicable when you have N independent events, each of which has a probability p of success, and a corresponding probability of q $(= 1 - p)$ of not succeeding. As an example, consider rolling a fair die 10 times. The binomial distribution helps answer questions such as "What is the probability that a 2 will show up exactly 4 times in the 10 rolls? (Here "success" in a trial means rolling a 2, $N = 10$, , $p = 1/6$, and $q = 5/6$.)

The *probability mass function* of the binomial distribution is

$$\begin{aligned} f_{N,p}(n) &= \begin{pmatrix} N \\ n \end{pmatrix} p^n (1-p)^{N-n} \\ &= \begin{pmatrix} N \\ n \end{pmatrix} p^n q^{N-n}, \end{aligned} \tag{1}$$

where

$$\begin{pmatrix} N \\ n \end{pmatrix} = \frac{N!}{n!(N-n)!} \tag{2}$$

is the *binomial coefficient*. In Eq. (1) the factor $p^n q^{N-n}$ gives the probability that the n successes occur in a specific set of events, (e.g., the probabilty that the 2's showed up in rolls, 1, 4, 5, and 10) and the binomial coefficient gives the number of possible sequences of events in which the successes could occur.

In the die-rolling example, the probability of getting a 2 on 5 of the rolls is

$$\begin{aligned} f_{10,1/6}(5) &= \begin{pmatrix} 10 \\ 5 \end{pmatrix} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^6 \\ &= \frac{10!}{4!6!} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^6 \\ &= 0.054, \end{aligned} \tag{3}$$

or a little more than 5%.

The mean of the binomial distribution is

$$\bar{n} = np, \tag{4}$$

and the standard deviation is

$$\sigma = \sqrt{npq}. \tag{5}$$

Now consider an experiment in which the number of events occurring in some time interval are counted, such as the number of radioactive decays recorded by a detector in some time T. It assumed that the decays are independent, i.e., that the occurence of a decay does not affect the probability of subsequent decays. Let's consider as an example an experiment in which the average number of detected decays λ is known, but the distribution of detected values is unknown. (What is the probability of detecting no decays? exactly one decay? λ decays? etc.)

We can, in principle, use the binomial distribution to answer this question. Break the time interval T into N sub-intervals, where N is large enough to make the detection of 2 or more decays in any sub-interval vanishingly small. Each sub-interval can then be considered an independent trial, with a "success" being the detection of a decay with probability λ/N, and "failure" being no detection of a decay. As an example, consider an experiment in which the average number of decays is $\lambda = 1$, and for our analysis we break the time interval into 50 sub-intervals. (This value of $N = 50$ is not really large enough to give exact results, but it's easy to work with, and the results are pretty good.)

The probability of detecting 0, 1, and 2 decays are thus

$$\begin{aligned} f_{50,1/50}(0) &\simeq \begin{pmatrix} 50 \\ 0 \end{pmatrix} \left(\frac{1}{50}\right)^0 \left(\frac{49}{50}\right)^{50} \\ &= 0.364 \qquad (6) \\ f_{50,1/50}(1) &\simeq \begin{pmatrix} 50 \\ 1 \end{pmatrix} \left(\frac{1}{50}\right)^1 \left(\frac{49}{50}\right)^{49} \\ &= 0.372 \qquad (7) \\ f_{50,1/50}(2) &\simeq \begin{pmatrix} 50 \\ 2 \end{pmatrix} \left(\frac{1}{50}\right)^2 \left(\frac{49}{50}\right)^{48} \\ &= 0.186 \qquad (8) \end{aligned}$$

As λ gets larger we need more sub-intervals for this scheme to work, and as N gets larger the factorials quickly become intractable. To get around this problem we turn to math, and take the large N limit of the binomial distribution function $f_{N,\lambda/N}$ (while keeping λ fixed). The result is the Poisson probability distribution given in Hughes and Hase Eq. (3.11):

$$P(n;\lambda) = \frac{e^{-\lambda}\lambda^n}{n!}. \tag{9}$$

The Poisson distribution gives the following results for the detection of 0, 1, and 2 decays in the previously considered example with $\lambda = 1$:

$$\begin{aligned} P(0, \lambda = 1) &= \frac{e^{-1}1^0}{0!} \\ &= 0.367 \qquad (10) \\ P(1, \lambda = 1) &= \frac{e^{-1}1^1}{1!} \\ &= 0.367 \qquad (11) \\ P(2, \lambda = 1) &= \frac{e^{-1}1^2}{2!} \\ &= 0.184 \qquad (12) \end{aligned}$$