Signals & Systems – Handout #2
Complex Numbers – A Review

H-2.1 Axiomatic Definition of Complex Numbers:

A *complex* number z is defined as an ordered pair (or vector) $\langle a,b\rangle$ of two arbitrary *real* numbers $a \in \mathbb{R}$ and $b \in \mathbb{R}$. We use the symbol "$\equiv$" to denote the *equivalency* between a complex number symbol z and its explicit components $a \in \mathbb{R}$ and $b \in \mathbb{R}$, i.e. $z \equiv \langle a,b\rangle$. We use the symbol $\mathbb{C}$ to denote the set of *all* complex numbers.

H-2.1.1 Equality:

Two complex numbers $z_1 \equiv \langle a_1,b_1\rangle$ and $z_2 \equiv \langle a_2,b_2\rangle$ are *equal,* i.e. $z_1 = z_2$, if and only if $a_1 = a_2$ and $b_1 = b_2$.

H-2.1.2 Addition:

The *sum* of two complex numbers $z_1 \equiv \langle a_1,b_1\rangle$ and $z_2 \equiv \langle a_2,b_2\rangle$ is defined as $z_1 + z_2 \equiv \langle\, a_1 + a_2\,,\, b_1 + b_2\,\rangle$.

H-2.1.3 Product:

The *product* of two complex numbers $z_1 \equiv \langle a_1,b_1\rangle$ and $z_2 \equiv \langle a_2,b_2\rangle$ is defined as $z_1 \cdot z_2 \equiv \langle\, a_1 a_2 - b_1 b_2\,,\, a_1 b_2 + b_1 a_2\,\rangle$.

H-2.1.4 Identity Elements:

We define the complex number "1" as the element $\langle 1,0\rangle$ and the complex number "0" as the element $\langle 0,0\rangle$. (Note that there is a notational ambiguity in using the symbols 1 and 0. The elements "0"$\in \mathbb{C}$ and "1"$\in \mathbb{C}$ are technically different from the elements $0 \in \mathbb{R}$ and $1 \in \mathbb{R}$. The ambiguity can be justified by embedding $\mathbb{R}$ in $\mathbb{C}$ as described in the following sections.)

H-2.2 Elementary Properties of Complex Numbers:

One can show that the addition, product, and identity elements defined in the previous section satisfy the requirements for a *field.* Namely, if z_1, z_2, z_3 are elements of $\mathbb{C}$ then:

(i) $z_1 + z_2$ and $z_1 \cdot z_2$ are also elements of $\mathbb{C}$ (closure).

(ii) $z_1 + z_2 = z_2 + z_1$ and $z_1 \cdot z_2 = z_2 \cdot z_1$ (commutativity).

(iii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ and $z_1\,(z_2\,z_3) = (z_1\,z_2)\,z_3$ (associativity).

(iv) $z_1\,(z_2 + z_3) = z_1\,z_2 + z_1\,z_3$ (distributivity).

(v) $z_1 + 0 = 0 + z_1 = z_1$ and $z_1 \cdot 1 = 1 \cdot z_1 = z_1$ (existence of identity).

(vi) for any z_1 there is a unique element denoted by $-z_1$ (with $(-z_1) \in \mathbb{C}$) such that $z_1 + (-z_1) = 0$ (inverse to addition).

(vii) for any $z_1 \neq 0$ there is a unique element denoted by z_1^{-1} (with $z_1^{-1} \in \mathbb{C}$) such that $z_1 \cdot (z_1^{-1}) = 1$ (inverse to multiplication).

H-2.2.1 Negative and Inverse Complex Numbers:

In light of items (vi) and (vii) we define the negative $-z$ of a complex number $z \equiv \langle a, b\rangle$ as $-z \equiv \langle -a, -b\rangle$ and the inverse z^{-1} of a complex number $z \equiv \langle a, b\rangle$ as $z^{-1} \equiv \langle \frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\rangle$.

H-2.2.2 Subtraction and Division of Complex Numbers:

We define the subtraction of two complex numbers z_1 and z_2 as $z_1 - z_2 = z_1 + (-z_2)$ and their division as $z_1/z_2 = z_1 \cdot (z_2^{-1})$.

H-2.3 Embedding $\mathbb{R}$ in $\mathbb{C}$:

Note that for any two complex numbers $z_1 \equiv \langle a_1, 0\rangle$ and $z_2 \equiv \langle a_2, 0\rangle$ we have $z_1 + z_2 \equiv \langle a_1 + a_2, 0\rangle$ and $z_1 \cdot z_2 \equiv \langle a_1 \cdot a_2, 0\rangle$. If we define that a *real* number α is equivalent to the special *complex* number $\langle \alpha, 0\rangle$ then we can compute the sum and product of two *real* numbers a_1 and a_2 by means of the *complex* operations "+" and "·". As such, we can *compatibly* embed any real number from $\mathbb{R}$ into the set of complex numbers $\mathbb{C}$, i.e.

$$\alpha \in \mathbb{R} \quad \leftrightarrow \quad \langle \alpha, 0\rangle \in \mathbb{C} \quad \Rightarrow \quad \mathbb{R} \subset \mathbb{C}.$$

Note, furthermore, that the same compatibility applies to the identity elements 1 in $\mathbb{R}$ (i.e. identity element "1"$\equiv \langle 1, 0\rangle$ in $\mathbb{C}$) and 0 in $\mathbb{R}$ (i.e. "0"$\equiv \langle 0, 0\rangle$ in $\mathbb{C}$).

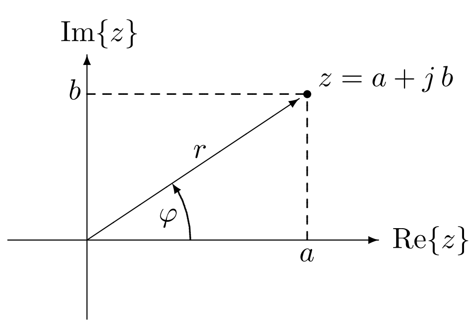
H-2.3.1 The Square Root of -1:

There is no number $a \in \mathbb{R}$ such that $a \cdot a$ equals -1. There is the number $j \equiv \langle 0, 1\rangle$ in $\mathbb{C}$, however, such that $j \cdot j \equiv \langle -1, 0\rangle$, which represents the real number -1 (according to the discussion of the previous section). The complex number j is thus frequently (and somewhat casually) written as $j = \sqrt{-1}$. Symbol j enables us to simplify the notation of complex numbers $\langle a, b\rangle$ as $a + j \cdot b \equiv \langle a, 0\rangle + \langle 0, 1\rangle \cdot \langle b, 0\rangle = \langle a, b\rangle$ with $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

H-2.3.2 The Real Part and the Imaginary Part of a Complex Number:

The functions *real part* and *imaginary part* of a complex number $z \equiv \langle a, b\rangle$ are defined as: $\mathrm{Re}\{z\} = a \in \mathbb{R}$ and $\mathrm{Im}\{z\} = b \in \mathbb{R} \quad \Rightarrow \quad z = \mathrm{Re}\{z\} + j \cdot \mathrm{Im}\{z\}$.

H-2.4 The Geometric Interpretation of a Complex Number:



With $z = a + j\,b \equiv \langle a, b\rangle$ for $a \in \mathbb{R}$ and $b \in \mathbb{R}$ we define:

(i) the *magnitude* of z as

$$|\,z\,| = r = \sqrt{a^2 + b^2}, \quad \text{and}$$

(ii) the *phase* of z (written $\varphi = \measuredangle\, z$) such that

$$z = r\,(\cos(\varphi) + j \sin(\varphi)).$$

H-2.5 Elementary Complex Number Calculations:

We assume below that the complex number z is given by $z = a + j\,b$ with $a \in \mathbb{R}$ and $b \in \mathbb{R}$. Similarly, we assume that $z_i = a_i + j\,b_i$ with $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}$ for $i = 1, 2, \ldots$ and so forth.

H-2.5.1 Addition, Subtraction, Multiplication, and Division:

$$z_1 + z_2 = (a_1 + a_2) + j\,(b_1 + b_2) \qquad z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + j\,(b_1 a_2 + a_1 b_2)$$

$$z_1 - z_2 = (a_1 - a_2) + j\,(b_1 - b_2) \qquad \frac{z_1}{z_2} = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) + j\left(\frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2}\right)$$

H-2.5.2 Conjugate Complex Numbers:

Definition: $z^* = a - j\,b = \mathrm{Re}\{z\} - j\;\mathrm{Im}\{z\}$

Elementary operations:

$$(z_1 + z_2)^* = z_1^* + z_2^* \qquad (z_1 - z_2)^* = z_1^* - z_2^*$$

$$(z_1 \cdot z_2)^* = z_1^* \cdot z_2^* \qquad (z_1/z_2)^* = z_1^*/z_2^*$$

$$\left(\sum_{i=1}^n z_i\right)^* = \sum_{i=1}^n z_i^* \qquad \left(\prod_{i=1}^n z_i\right)^* = \prod_{i=1}^n z_i^*$$

H-2.5.3 Absolute Value and Magnitude Squared:

Definition: $|\,z\,|^2 = z \cdot z^* = a^2 + b^2 = (\mathrm{Re}\{z\})^2 + (\mathrm{Im}\{z\})^2$

Elementary operations:

$$|\,z_1 + z_2\,| \le |\,z_1\,| + |\,z_2\,| \qquad |\,z_1 - z_2\,| \ge |\,z_1\,| - |\,z_2\,|$$

$$|\,z_1 \cdot z_2\,| = |\,z_1\,| \cdot |\,z_2\,| \qquad |\,z_1/z_2\,| = |\,z_1\,| \,/\, |\,z_2\,|$$

$$\left|\sum_{i=1}^n z_i\right| \le \sum_{i=1}^n |\,z_i\,| \qquad \left|\prod_{i=1}^n z_i\right| = \prod_{i=1}^n |\,z_i\,|$$

also $\quad |\,z_1\,| - |\,z_2\,| \le |\,z_1 + z_2\,|$

H-2.5.4 Real Part and Imaginary Part:

$$\mathrm{Re}\{z\} = a = \tfrac{1}{2}(z + z^*) = |\,z\,| \cdot \cos(\measuredangle z) \qquad \mathrm{Im}\{z\} = b = \tfrac{1}{2j}(z - z^*) = |\,z\,| \cdot \sin(\measuredangle z)$$

H-2.5.5 Phase Angles:

The phase $\measuredangle z$ of a complex number $z \neq 0$ can be computed as follows (note that the phase of the number $z = 0$ is *not* defined):

Closed form formula: $\measuredangle z = \tan^{-1}(\tfrac{b}{a}) + \tfrac{\pi}{2}\left[\mathrm{sign}(b) - \mathrm{sign}(\tfrac{b}{a})\right]$ for $a \neq 0$

and $\measuredangle z = \tfrac{\pi}{2}\,\mathrm{sign}(b)$ for $a = 0$.

Case-by-case formula:

$$\measuredangle z = \begin{cases} \tan^{-1}(\tfrac{b}{a}) & \text{if } a > 0 \\ \tfrac{\pi}{2} \cdot \mathrm{sign}(b) & \text{if } a = 0 \\ \tan^{-1}(\tfrac{b}{a}) + \pi \cdot \mathrm{sign}(b) & \text{if } a < 0. \end{cases}$$

Note that $\tan^{-1}(x)$ refers to the *main branch* inverse function of $\tan(x)$, i.e. $-\frac{\pi}{2} \le \tan^{-1}(x) \le \frac{\pi}{2}$ for all $x \in \mathbb{R}$. The formulas above are normalized such that $-\pi \le \measuredangle z \le +\pi$. We always have $\tan(\measuredangle z) = b/a$. All angles are in radians.

H-2.5.6 (Integer) Powers of Complex Numbers:

De Moivre's Theorem: $z^n = |\,z\,|^n \left[\cos(n \cdot \measuredangle z) + j\,\sin(n \cdot \measuredangle z)\right]$ for all $n \in \mathbb{Z}$.

(Note: De Moivre's Theorem remains generally true for powers n other than integer powers as well.)

H-2.6 Complex Functions:

H-2.6.1 Functions of the Form $\mathbb{R} \rightarrow \mathbb{C}$ (Type I):

We define a Type I complex function $z(t)$ as a function whose *domain* is (a subset of) $\mathbb{R}$ and whose *range* is (a subset of) $\mathbb{C}$. Type I functions can be decomposed into two component functions $a(t) : \mathbb{R} \rightarrow \mathbb{R}$ and $b(t) : \mathbb{R} \rightarrow \mathbb{R}$ such that $z(t) = a(t) + j\, b(t)$.

Continuity – A Type I complex function $z(t)$ is continuous if its components $a(t)$ and $b(t)$ are continuous.

Differentiability – A Type I complex function $z(t)$ is differentiable with respect to t if its components $a(t)$ and $b(t)$ are differentiable, i.e.

$$\tfrac{d}{dt}\, z(t) = \tfrac{d}{dt}\, a(t) + j\, \tfrac{d}{dt}\, b(t).$$

Integrability – A Type I complex function $z(t)$ is integrable with respect to t if its components $a(t)$ and $b(t)$ are integrable, i.e.

$$\int_{t_0}^{t_1} z(t)\, dt = \int_{t_0}^{t_1} a(t)\, dt + j \cdot \int_{t_0}^{t_1} b(t)\, dt.$$

Graphic Representation – A continuous Type I complex function $z(t)$ describes a *contour* (i.e. a *continuous curve*) in the complex plane ($\text{Im}\{z(t)\}$ vs. $\text{Re}\{z(t)\}$) as variable/parameter t traverses $\mathbb{R}$.

H-2.6.2 Functions of the Form $\mathbb{C} \rightarrow \mathbb{C}$ (Type II):

We define a Type II complex function $f(z)$ as a function whose *domain* is (a subset of) $\mathbb{C}$ and whose *range* is (a subset of) $\mathbb{C}$. Type II functions can be decomposed into the component functions $a(\alpha, \beta) : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $b(\alpha, \beta) : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $f(z) = a(\,\text{Re}\{z\}, \text{Im}\{z\}\,) + j\, b(\,\text{Re}\{z\}, \text{Im}\{z\}\,)$.

Differentiability – A Type II complex function $f(z)$ is differentiable with respect to z if its components $a(\alpha, \beta)$ and $b(\alpha, \beta)$ satisfy the Cauchy-Riemann differential equations:

$$\tfrac{\partial}{\partial \alpha}\, a(\alpha, \beta) = \tfrac{\partial}{\partial \beta}\, b(\alpha, \beta) \quad \text{and} \quad \tfrac{\partial}{\partial \beta}\, a(\alpha, \beta) = -\tfrac{\partial}{\partial \alpha}\, b(\alpha, \beta).$$

For convenience we write $f'(z)$ to denote $\tfrac{d}{dz}\, f(z)$. Similarly we define $f''(z) = \tfrac{d}{dz}\, f'(z)$, $f'''(z) = \tfrac{d}{dz}\, f''(z)$, $\ldots$, $f^{(n+1)}(z) = \tfrac{d}{dz}\, f^{(n)}(z)$, and so forth.

Contour Integral – A Type II complex function $f(z)$ may be integrated along a *contour* $\mathcal{C}$ defined by a continuous Type I function $z(t)$ for $t \in [t_0, t_1]$. If $z(t)$ is differentiable then we can convert the Type II integral into a Type I integral:

$$\int_{\mathcal{C}} f(z)\, dz = \int_{t_0}^{t_1} f(z(t)) \cdot [\, \tfrac{d}{dt}\, z(t)\,]\, dt.$$

Analytic Functions – If a Type II complex function $f(z)$ is differentiable for *every* $z \in \mathbb{C}$ then it is called *analytic* over $\mathbb{C}$. *Analytic functions* have well defined derivatives of *any* order, i.e. not only the first derivative exists, but also the second, third, forth, and so on for any arbitrary order.

Path Independent Integration – If a Type II complex function $f(z)$ is *analytic* over $\mathbb{C}$ then the value of every contour integral over $f(z)$ becomes *path independent,* i.e. it only depends on the start point and end point of the integration. As a result we can define a function $F(z) = \int_{z_0}^{z} f(\xi)\, d\xi$ for some arbitrary z_0 such that $\int_{z_1}^{z_2} f(z)\, dz = F(z_2) - F(z_1)$ is independent of the path between z_1 and z_2.

Taylor Series – A Type II complex function $f(z)$ that is *analytic* over $\mathbb{C}$ may be expanded in a *Taylor series*:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, z^n \quad \text{for all} \quad z \in \mathbb{C}.$$

Important examples for such Taylor series expansions include:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad \cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

H-2.7 Euler's Identity:

With the Taylor series of e^z, $\sin(z)$, and $\cos(z)$ it is easy to prove that for any complex number z with $r = |\, z\,|$ and $\varphi = \measuredangle z$ we can write

$$z = r\, e^{j\,\varphi} = r\,(\,\cos(\varphi) + j\, \sin(\varphi)\,).$$

More generally, for arbitrary complex numbers z we have

$$e^{j\,z} = \cos(z) + j\, \sin(z) \qquad \cos(z) = \frac{1}{2}\,(e^{j\,z} + e^{-j\,z}) \qquad \sin(z) = \frac{1}{2j}\,(e^{j\,z} - e^{-j\,z}).$$

Analogously we define the complex *hyperbolic* functions

$$e^z = \cosh(z) + \sinh(z) \qquad \cosh(z) = \frac{1}{2}\,(e^z + e^{-z}) \qquad \sinh(z) = \frac{1}{2}\,(e^z - e^{-z}).$$

Manipulation rules for complex exponentials with two complex numbers z_1 and z_2:

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2} \qquad e^{z_1-z_2} = e^{z_1} / e^{z_2} \qquad (e^{z_1})^{\,z_2} = e^{z_1 \cdot z_2}.$$

H-2.8 Complex Logarithms and "Multiple-Valued" Functions:

From Euler's identity it is obvious that the phase $\varphi = \measuredangle z$ of a complex number in not unique since $z = r\, e^{j\,\varphi} = r\, e^{j\,(\varphi+2\pi k)}$ for any $k \in \mathbb{Z}$. The ambiguity is immaterial for operations that are explicitly defined on $\mathrm{Re}\{z\}$ and $\mathrm{Im}\{z\}$ (such as complex multiplication and complex addition for example). The ambiguity becomes problematic, though, for functions that operate explicitly on $r = |\, z\,|$ and $\varphi = \measuredangle z$ such as the complex logarithm

$$\ln(z) = \ln(r\, e^{j\,(\varphi+2\pi k)}) = \ln(r) + \ln(e^{j\,(\varphi+2\pi k)}) = \ln(r) + j\,(\varphi + 2\pi k) \quad \text{for all} \quad k \in \mathbb{Z}.$$

Functions such as the complex logarithm are casually called "multiple valued" functions (even though a *function* may by definition *not* be multiple valued!). One way to address the conundrum of "multiple valued" functions is to limit such functions to their *main branch,* e.g. chose k such that $-\pi < \mathrm{Im}\{\ln(z)\} \leq +\pi$.

Manipulation rules for complex logarithms with $z_1, z_2 \in \mathbb{C}$:

$$\ln(z_1 \cdot z_2) = \ln(z_1) + \ln(z_2) \qquad \ln(z_1/\, z_2) = \ln(z_1) - \ln(z_2) \qquad \ln(z_1^{z_2}) = z_2 \cdot \ln(z_1).$$

H-2.9 Complex Powers:

An operation that may also lead to ambiguous (i.e. "multiple valued") results is *complex exponentiation,* i.e. an operation of the form z^v with $z \in \mathbb{C}$ and $v \in \mathbb{C}$. With $z = r\, e^{j\,\varphi}$ ($r, \varphi \in \mathbb{R}$) and $v = \alpha + j\,\beta$ ($\alpha, \beta \in \mathbb{R}$) we define

$$z^v = e^{v\,\ln(z)} = e^{(\alpha + j\,\beta)\cdot(\ln(r) + j\,(\varphi + 2\pi k))} = e^{[\alpha\ln(r) - \beta\varphi]}\, e^{j[\alpha\varphi + \beta\ln(r)]}\, e^{-2\pi k[\beta - j\alpha]} \quad \text{for} \quad k \in \mathbb{Z}.$$

Note that the only case for which the ambiguity disappears is for $\beta = 0$ and $\alpha \in \mathbb{Z}$, i.e. for expressions of the form z^n with $n \in \mathbb{Z}$. In all other cases we may, again, restrict to the main branch of the complex logarithm $\ln(z)$.

Manipulation rules for complex powers with $z, v, z_1, z_2, v_1, v_2 \in \mathbb{C}$:

$$z^{v_1+v_2} = z^{v_1} \cdot z^{v_2} \qquad z^{v_1-v_2} = z^{v_1}/\, z^{v_2} \qquad (z^{v_1})^{v_2} = z^{v_1\cdot v_2}$$

$$(z_1 \cdot z_2)^v = z_1^v \cdot z_2^v \qquad (z_1/\, z_2)^v = z_1^v/\, z_2^v.$$

H-2.10 The Roots of Unity:

The name n^{th}-*roots of unity* refers to the set of complex numbers z_k for $k = 0 \ldots n-1$ such that $(z_k)^n = 1$. One can show that the n^{th}-*roots of unity* are given by $z_k = e^{j2\pi(k/n)}$ for $k = 0 \ldots n-1$. We may write in polynomial form

$$z^n - 1 = \prod_{k=0}^{n-1}(z - z_k) \quad \text{with} \quad z_k = e^{j\pi((2k)/n)} \quad \text{for} \quad k = 0 \ldots n-1.$$

Alternatively, we may study the case $(z_k)^n = -1$ for $k = 0 \ldots n-1$ which leads to

$$z^n + 1 = \prod_{k=0}^{n-1}(z - z_k) \quad \text{with} \quad z_k = e^{j\pi((2k+1)/n)} \quad \text{for} \quad k = 0 \ldots n-1.$$

H-2.11 The Fundamental Theorem of Algebra:

Consider a set of $n+1$ complex numbers $a_i \in \mathbb{C}$ for $i = 0 \ldots n$ and the associated complex polynomial $P(z) = a_0 + \sum_{i=1}^{n} a_i\, z^n$. One can show that there exists a unique set of n complex numbers z_i for $i = 1 \ldots n$ such that $P(z_i) = 0$ for $i = 1 \ldots n$ and such that

$$P(z) = a_0 + \sum_{i=1}^{n} a_i\, z^n = a_n \cdot \prod_{i=1}^{n}(z - z_i).$$