Signals & Systems Handout #3

H-3.1 Elementary Continuous-Domain Functions:

Continuous-domain functions are defined for $t \in \mathbb{R}$.

H-3.1.1 Step-Function:

$$\mu(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1/2 & \text{for} \quad t = 0 \\ 1 & \text{for} \quad t > 0 \end{cases}$$

H-3.1.2 Rect-Function:

$$\mathrm{rect}(t) = \begin{cases} 0 & \text{for} \quad |\,t\,| > 1/2 \\ 1/2 & \text{for} \quad |\,t\,| = 1/2 \\ 1 & \text{for} \quad |\,t\,| < 1/2 \end{cases}$$

H-3.1.3 Sign-Function:

$$\mathrm{sign}(t) = \begin{cases} -1 & \text{for} \quad t < 0 \\ 0 & \text{for} \quad t = 0 \\ +1 & \text{for} \quad t > 0 \end{cases}$$

H-3.1.4 Triangle-Function:

$$\Delta(t) = \begin{cases} 0 & \text{for} \quad |\,t\,| \geq 1 \\ 1 - |\,t\,| & \text{for} \quad |\,t\,| < 1 \end{cases}$$

H-3.1.5 Sinc-Function:

$$\mathrm{sinc}(t) = \begin{cases} \frac{\sin(t)}{t} & \text{for} \quad t \neq 0 \\ 1 & \text{for} \quad t = 0 \end{cases}$$

H-3.2 Classification of Continuous-Domain Signals:

We consider continuous-domain signals $x(t)$ that are defined for $t \in \mathbb{R}$. The range of continuous-domain signals may be real ($x(t) \in \mathbb{R}$) or complex ($x(t) \in \mathbb{C}$).

H-3.2.1 Periodic Signals:

A continuous-domain signal $x(t)$ is *periodic* with *period* τ if there is a $\tau \in \mathbb{R}$ such that $x(t) = x(t - \tau)$ for all $t \in \mathbb{R}$.

H-3.2.2 Symmetric Signals:

A continuous-domain signal $x(t)$ is of *even symmetry* if $x(t) = x(-t)$. It is of *odd symmetry* if $x(t) = -x(-t)$. A (complex-valued) signal is of *even Hermitian symmetry* if $x(t) = x^*(-t)$. It is of *odd Hermitian symmetry* if $x(t) = -x^*(-t)$.

H-3.2.3 Bounded Signals:

A continuous-domain signal $x(t)$ is *bounded* if $|\,x(t)\,| \leq \mathcal{B}_x < \infty$ for some finite $\mathcal{B}_x \in \mathbb{R}^+$. (In writing $\mathcal{B}_x$ we imply the smallest number such that $|\,x(t)\,| \leq \mathcal{B}_x$.)

H-3.2.4 Energy Signals:

A continuous-domain signal $x(t)$ is an *energy signal* if its energy $\mathcal{E}_x$ is finite.

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |\,x(t)\,|^2 \, dt < \infty$$

H-3.2.5 Power Signals:

A continuous-domain signal $x(t)$ is a *power signal* if its power $\mathcal{P}_x$ is finite.

$$\mathcal{P}_x = \lim_{\vartheta \to \infty} \tfrac{1}{2\,\vartheta} \int_{-\vartheta}^{\vartheta} |\,x(t)\,|^2 \, dt < \infty$$

A periodic signal $x(t)$ with period τ is a power signal with $\mathcal{P}_x = \frac{1}{\tau} \int_0^{\tau} |\,x(t)\,|^2 \, dt$.

H-3.2.6 Absolutely Integrable Signals:

A continuous-domain signal $x(t)$ is *absolutely integrable* if

$$\mathcal{S}_x = \int_{-\infty}^{\infty} |\,x(t)\,| \, dt < \infty.$$

H-3.2.7 Finite Length Signals:

A continuous-domain signal $x(t)$ is of *finite length* if there exists a t_1 and a t_2 with $t_1 \leq t_2$ such that $x(t) = 0$ for all $t < t_1$ and $t > t_2$. Let $\tilde{t}_1$ denote the largest possible t_1 such that $x(t) = 0$ for all $t < \tilde{t}_1$ and let $\tilde{t}_2$ denote the smallest possible t_2 such that $x(t) = 0$ for all $t > \tilde{t}_2$ then the length of $x(t)$ is defined by:

$$\mathcal{L}_x = \tilde{t}_2 - \tilde{t}_1.$$

Note that the length of a signal $x(t)$ that is identically equal to zero for all $t \in \mathbb{R}$ is not defined!

H-3.2.8 Causal and Anti-Causal Signals:

A continuous-domain signal $x(t)$ is *causal* if $x(t) = 0$ for all $t < 0$. It is *anti-causal* if $x(t) = 0$ for all $t > 0$.

H-3.3 Elementary Continuous-Domain Signal Operations:

H-3.3.1 Convolution:

The continuous-domain *convolution* of two signals $x(t)$ and $h(t)$ is defined by

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau.$$

Convolution generally involves folding, shifting, multiplication, and integration.

H-3.3.2 Properties of Convolution:

The continuous-domain *convolution* operator $*$ has the following properties:

a)	Commutativity:	$x(t) * h(t) = h(t) * x(t)$
b)	Distributivity:	$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$
c)	Associativity:	$(x_1(t) * x_2(t)) * x_3(t) = x_1(t) * (x_2(t) * x_3(t))$
d)	Shift Property:	$h(t) * x(t) = y(t) \;\Rightarrow\; h(t-\tau) * x(t) = y(t-\tau)$
e)	Convolution Length:	$y(t) = h(t) * x(t) \;\Rightarrow\; \mathcal{L}_y \leq \mathcal{L}_h + \mathcal{L}_x$

H-3.3.3 Elementary Convolution Identities:

a) $\mathrm{rect}(t) * \mathrm{rect}(t) = \Delta(t)$
b) $\mu(t) * \mu(t) = t\,\mu(t)$
c) $(e^{\lambda t}\mu(t)) * \mu(t) = \frac{e^{\lambda t}-1}{\lambda}\,\mu(t)$

H-3.3.4 Deterministic Correlation:

The *(deterministic) correlation* of two energy signals $x(t)$ and $y(t)$ is defined by

$$\mathcal{R}_{xy}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\,y^*(t)\,dt = x(\tau) * y^*(-\tau).$$

For two power signals $x(t)$ and $y(t)$ we define respectively

$$\tilde{\mathcal{R}}_{xy}(\tau) = \lim_{\vartheta \to \infty} \tfrac{1}{2\vartheta} \int_{-\vartheta}^{\vartheta} x(t+\tau)\,y^*(t)\,dt.$$

For two signals $x(t)$ and $y(t)$ that are both periodic with period τ we obtain

$$\tilde{\mathcal{R}}_{xy}(\tau) = \frac{1}{\tau} \int_{0}^{\tau} x(t+\tau)\,y^*(t)\,dt.$$

H-3.4 The Dirac Impulse Function:

H-3.4.1 "Casual" Definition of the Dirac Impulse:

A "casual" definition of the *Dirac impulse* $\delta(t)$ is provided by a limiting argument. Consider an infinitely narrow yet infinitely high impulse which is constructed such that the area underneath the impulse has a value of one:

$$\delta(t) \text{ is represented by "} \lim_{\tau \to 0} \tfrac{1}{\tau}\,\mathrm{rect}(\tfrac{t}{\tau})\text{."}$$

H-3.4.2 Rigorous Definition of the Dirac Impulse:

A rigorous definition of the *Dirac impulse* $\delta(t)$ is beyond the scope of this course and requires the study of the *space of linear functionals* $\mathbb{S}$ over a suitably chosen *space of testing functions* $\mathbb{D}$. An expression of the form $\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = x(0)$ can be viewed as a linear mapping of $x(t) \in \mathbb{D}$ into the scalar value $x(0) \in \mathbb{C}$. The *Dirac impulse* $\delta(t) \in \mathbb{S}$ becomes a symbolic notation for the mapping: $x(t) \in \mathbb{D} \to x(0) \in \mathbb{C}$. Operations that involve $\delta(t)$ are mathematically meaningful only over the backdrop of space $\mathbb{S}$.

H-3.4.3 Properties of the Dirac Impulse:

a) Area: $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$
b) Sampling: $\int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\,dt = x(\tau)$
c) Exchange: $x(t)\,\delta(t-\tau) = x(\tau)\,\delta(t-\tau)$
d) Scaling: $\delta(\alpha\,t) = \frac{1}{|\alpha|}\,\delta(t)$
e) Convolution: $x(t) * \delta(t-\tau) = x(t-\tau)$
f) Symmetry: $\delta(t) = \delta(-t)$

H-3.5 Classification of Continuous-Domain Systems:

We consider continuous-domain systems $\mathfrak{T}$ with input $x(t)$ and output $y(t)$.

$$y(t) = \mathfrak{T}\{\, x(t) \,\}$$

H-3.5.1 Linear Systems:

A continuous-domain system $\mathfrak{T}$ is *linear* if for any two arbitrary input signals $x_1(t)$, $x_2(t)$ and for any two constants α_1, $\alpha_2 \in \mathbb{R}$ (or $\mathbb{C}$) we have

$$\mathfrak{T}\{\, \alpha_1\, x_1(t) + \alpha_2\, x_2(t) \,\} = \alpha_1\, \mathfrak{T}\{\, x_1(t) \,\} + \alpha_2\, \mathfrak{T}\{\, x_2(t) \,\}.$$

H-3.5.2 Time-Invariant Systems:

A continuous-domain system $\mathfrak{T}$ is *time-invariant* if $y(t) = \mathfrak{T}\{\, x(t) \,\}$ implies that $y(t-\tau) = \mathfrak{T}\{\, x(t-\tau) \,\}$ for any arbitrary input signal $x(t)$ any arbitrary delay $\tau \in \mathbb{R}$.

H-3.5.3 Causal Systems:

A continuous-domain system $\mathfrak{T}$ is *causal* if the output $y(t)$ at time t only depends on current and past input values $x(\vartheta)$ for $\vartheta \le t$ and/or only depends on past output values $y(\vartheta)$ for $\vartheta < t$.

H-3.5.4 BIBO Stable Systems:

A continuous-domain system $\mathfrak{T}$ is *bounded-input bounded-output (BIBO) stable* if any bounded input $|x(t)| \le \mathcal{B}_x < \infty$ leads to a bounded output $|y(t)| \le \mathcal{B}_y < \infty$.

H-3.6 Continuous Linear Time-Invariant (CLTI) Systems:

H-3.6.1 Impulse Response:

Let $\mathfrak{T}$ denote a CLTI system. If we let the *impulse response* $h(t)$ of $\mathfrak{T}$ be defined as $h(t) = \mathfrak{T}\{\, \delta(t) \,\}$ then the response of $\mathfrak{T}$ to an arbitrary input $x(t)$ is given by

$$y(t) = x(t) * h(t).$$

H-3.6.2 Causal CLTI Systems:

A CLTI system $\mathfrak{T}$ is *causal* if and only if its *impulse response* $h(t)$ is a causal signal:

$$h(t) = 0 \quad \text{for} \quad t < 0.$$

H-3.6.3 BIBO Stable CLTI Systems:

A CLTI system $\mathfrak{T}$ is *BIBO stable* if and only if its *impulse response* $h(t)$ is absolutely integrable, i.e. if $\mathcal{S}_h < \infty$.

H-3.6.4 Eigenfunctions of CLTI Systems:

Input functions of the form $x(t) = e^{s_0 t}$ are *eigenfunctions* of CLTI systems.

$$y(t) = \mathfrak{T}\{\, e^{s_0 t} \,\} = h(t) * e^{s_0 t} = e^{s_0 t} \cdot \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s_0 \tau}\, d\tau}_{= H(s_0)} = e^{s_0 t} \cdot H(s_0)$$

When passed through a CLTI system, these eigenfunctions remain unchanged up to a constant (possibly complex) gain $H(s_0)$.

H-3.7 The Laplace Transform:

H-3.7.1 Definition of the (Bilateral) Laplace Transform:

The *(bilateral) Laplace transform* $X(s)$ of signal $x(t)$ is defined by

$$X(s) = \mathfrak{L}\{\, x(t) \,\} = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$
$$\text{with} \quad \text{ROC:}\ -\infty \le \sigma_1 < \text{Re}\{\, s \,\} < \sigma_2 \le +\infty.$$

The *Laplace transform* always consists of both the complex function $X(s)$ and its associated *region of convergence* (ROC). The region of convergence is the set of all complex values s for which the transform integral converges. The ROC is generally a vertical strip in the complex plane that extends from $-j\,\infty$ to $+j\,\infty$ and is bounded on the real axis between $\sigma_1 \in \mathbb{R}$ and $\sigma_2 \in \mathbb{R}$ (with $\sigma_1 < \sigma_2$). The constant $\sigma_1 \in \mathbb{R}$ is determined by the rate of exponential increase/decrease of the causal part of $x(t)$. Similarly, $\sigma_2 \in \mathbb{R}$ is determined by the rate of exponential increase/decrease of the anti-causal part of $x(t)$.

H-3.7.2 The Inverse Laplace Transform:

The *inverse Laplace transform* is defined as

$$x(t) = \mathfrak{L}^{-1}\{\, X(s) \,\} = \frac{1}{2\,\pi\, j} \int_{\mathcal{C}} X(s)\, e^{st}\, ds$$

in which the integration contour $\mathcal{C}$ is given by

$$\mathcal{C}\colon \sigma + j\,\Omega \left|\begin{matrix} \Omega = +\infty \\ \\ \Omega = -\infty \end{matrix}\right. \qquad \text{for some fixed} \quad \sigma \in \,]\, \sigma_1, \sigma_2 \,[.$$

H-3.7.3 Complex Contour Integration:

If a sufficiently smooth complex contour $\mathcal{C}$ can be described with a parameter description $p(\varphi) \in \mathbb{C}$ for $\varphi \in [\,a, b\,]$ then $\int_{\mathcal{C}} F(s)\, ds = \int_a^b F(p(\varphi))\, p'(\varphi)\, d\varphi$. A complex contour integral can thus be reduced to a conventional Riemann integral.

H-3.7.4 Two Elementary Laplace Transform Identities ($\lambda \in \mathbb{R}$):

$x(t) = \mathfrak{L}^{-1}\{ X(s) \}$	$X(s) = \mathfrak{L}\{ x(t) \}$	ROC:
$e^{-\lambda t}\,\mu(t)$	$\frac{1}{s+\lambda}$	$\mathrm{Re}\{ s \} > -\lambda$
$-e^{-\lambda t}\,\mu(-t)$	$\frac{1}{s+\lambda}$	$\mathrm{Re}\{ s \} < -\lambda$

H-3.7.5 The Laplace Transform of Causal Signals:

Note that every valid Laplace transform expression $X(s)$ has only *one* causal inverse transform $x(t)$. We do not need to know the ROC explicitly to find the correct causal inverse of $X(s)$.

H-3.7.6 A Short Table of Laplace Transforms of Causal Signals ($\lambda \in \mathbb{R}$):

$x(t) = \mathfrak{L}^{-1}\{ X(s) \}$	$X(s) = \mathfrak{L}\{ x(t) \}$	ROC:
$\delta(t)$	1	$s \in \mathbb{C}$
$\mu(t)$	$\frac{1}{s}$	$\mathrm{Re}\{ s \} > 0$
$t^n\, e^{-\lambda t}\,\mu(t)$	$\frac{n!}{(s+\lambda)^{n+1}}$	$\mathrm{Re}\{ s \} > -\lambda$
$e^{-\lambda t}\,\cos(\Omega_0\, t)\,\mu(t)$	$\frac{s+\lambda}{(s+\lambda)^2+\Omega_0^2}$	$\mathrm{Re}\{ s \} > -\lambda$
$e^{-\lambda t}\,\sin(\Omega_0\, t)\,\mu(t)$	$\frac{\Omega_0}{(s+\lambda)^2+\Omega_0^2}$	$\mathrm{Re}\{ s \} > -\lambda$

H-3.7.7 Properties of the Bilateral Laplace Transform:

Operation	$x(t) = \mathfrak{L}^{-1}\{\, X(s) \,\}$	$X(s) = \mathfrak{L}\{\, x(t) \,\}$ and ROC[a]
Linearity	$\alpha_1\, x_1(t) + \alpha_2\, x_2(t)$	$\alpha_1\, X_1(s) + \alpha_2\, X_2(s)$ $\quad \mathrm{ROC}_1 \cap \mathrm{ROC}_2$
Differentiation in Time	$(\frac{d}{dt})^n\, x(t)$	$s^n\, X(s)$ and same ROC
Time Shift	$x(t-\tau)$	$X(s)\, e^{-s\,\tau}$ and same ROC
Modulation	$x(t)\, e^{s_0\, t}$	$X(s-s_0)$ ROC is shifted by $\mathrm{Re}\{\, s_0 \,\}$
Differentiation in Frequency	$-t\, x(t)$	$\frac{d}{ds}\, X(s)$ and same ROC
Convolution	$x(t) * h(t)$	$X(s) \cdot H(s)$ $\quad \mathrm{ROC}_1 \cap \mathrm{ROC}_2$

[a] The actual ROC of the result of an operation may be larger than the one provided in the table. Check the common literature on Laplace transforms for the details.

H-3.8 CLTI Systems and the Laplace Transform:

H-3.8.1 Transfer Functions and BIBO Stable Systems:

Let $H(s) = \mathfrak{L}\{\, h(t) \,\}$ denote the Laplace transform of the impulse response $h(t)$ of a CLTI system. $H(s)$ is called the *transfer function* of the CLTI system. A CLTI system is *BIBO stable* if the imaginary axis is contained in the ROC of its transfer function $H(s)$.

H-3.8.2 Linear Constant Coefficient Differential Equations:

Every *linear constant coefficient differential equation* with input $x(t)$ and output $y(t)$ establishes a *causal linear time-invariant system.*

$$
\begin{aligned}
y(t) = -a_1 \tfrac{d}{dt}\, y(t) - a_2\, (\tfrac{d}{dt})^2\, y(t) - \ldots \\
\ldots - a_N\, (\tfrac{d}{dt})^N\, y(t) + b_0\, x(t) + b_1 \tfrac{d}{dt}\, x(t) + b_2\, (\tfrac{d}{dt})^2\, x(t) + \ldots \\
\ldots + b_M\, (\tfrac{d}{dt})^M\, x(t)
\end{aligned}
$$

By transforming the differential equation into the Laplace domain we obtain its *transfer function* $H(s)$. The transfer function of a linear constant coefficient differential equation is rational in variable s:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_0 + b_1 s + b_2 s^2 + \ldots + b_M s^M}{1 + a_1 s + a_2 s^2 + \ldots + a_N s^N}$$

Since $H(s)$ is the transfer function of a *causal* system we do not need to explicitly provide its ROC. Furthermore, we can write every rational transfer function of the form above in terms of its poles p_i (for $i = 1 \ldots N$) and zeros z_i (for $i = 1 \ldots M$).

$$H(s) = G\,\frac{(s - z_1)(s - z_2)\ldots(s - z_M)}{(s - p_1)(s - p_2)\ldots(s - p_N)}$$

The term G is often referred to as the *gain* of the system. Note, however, that G is usually *not* equal to the DC gain or the high-frequency gain of a system!

H-3.8.3 Stability of Causal CLTI Systems with Rational Transfer Functions:

A causal CLTI system with a *rational* transfer function $H(s)$ is stable if and only if the real part of all of its poles is strictly smaller than zero, i.e. $\mathrm{Re}\{\, p_i \,\} < 0$ for $i = 1 \ldots N$.

H-3.8.4 System I/O Description in the Laplace Domain:

Due to the convolution theorem of the Laplace transform we can find the output $y(t)$ of a CLTI system for a given input $x(t)$ conveniently in the Laplace Domain:

$$Y(s) = \mathfrak{L}\{\, y(t) \,\} = H(s) \cdot X(s) = \mathfrak{L}\{\, h(t) \,\} \cdot \mathfrak{L}\{\, x(t) \,\}.$$

If $Y(s)$ is rational then we can find its inverse transform $y(t)$ via a partial fraction expansion and a table lookup.

H-3.9 The Fourier Transform:

H-3.9.1 Definition of the Fourier Transform:

The *continuous time Fourier transform* and its *inverse transform* are defined by

$$X(\Omega) = \mathfrak{F}\{\, x(t) \,\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\,\Omega\, t}\, dt$$

$$\text{and} \qquad x(t) = \mathfrak{F}^{-1}\{\, X(\Omega) \,\} = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)\, e^{j\,\Omega\, t}\, d\Omega$$

The existence of the Fourier transform is guaranteed for absolutely integrable signals. For other signals meaningful definitions for Fourier transforms may be found, but the existence is not guaranteed in general. Note that we can directly derive the Fourier transform $X(\Omega)$ of a signal $x(t)$ from its Laplace transform $X(s)$ if the ROC of $X(s)$ contains the imaginary axis.

$$X(\Omega) = X(s)\,|_{\,s=j\,\Omega} \quad \text{if} \quad j\,\Omega \in \text{ROC} \quad \text{for all} \quad \Omega \in \mathbb{R}$$

There is an ambiguity in our notation for the Laplace transform $X(s)$ and the Fourier transform $X(\Omega)$. The distinction is achieved with the name of the independent variable: (s) for the Laplace transform and (Ω) for the Fourier transform.

H-3.9.2 Some Elementary Fourier Transform Identities:

Type	$x(t) = \mathfrak{F}^{-1}\{\, X(\Omega)\,\}$	$X(\Omega) = \mathfrak{F}\{\, x(t)\,\}$
Constant	$\frac{C}{2\pi}$	$C\,\delta(\Omega)$
Impulse	$\delta(t-\tau)$	$e^{-j\,\Omega\,\tau}$
Complex Exponential	$e^{j\,\Omega_0\,t}$	$2\pi\delta(\Omega-\Omega_0)$
Cosine	$\cos(\Omega_0\,t)$	$\pi\,[\,\delta(\Omega+\Omega_0)+\delta(\Omega-\Omega_0)\,]$
Sine	$\sin(\Omega_0\,t)$	$j\,\pi\,[\,\delta(\Omega+\Omega_0)-\delta(\Omega-\Omega_0)\,]$
Step-Function	$\mu(t)$	$\pi\,\delta(\Omega)+\frac{1}{j\,\Omega}$
Exponential Impulse	$\frac{1}{\lvert\tau\rvert}\,e^{-\frac{t}{\lvert\tau\rvert}}\,\mu(t)$	$\frac{1}{1+j\,\Omega\,\lvert\tau\rvert}$
Two-Sided Exponential	$\frac{1}{2\lvert\tau\rvert}\,e^{-\frac{\lvert t\rvert}{\lvert\tau\rvert}}$	$\frac{1}{1+\lvert\Omega\,\tau\rvert^2}$
Gaussian Impulse	$e^{-\pi t^2}$	$e^{-\frac{\Omega^2}{4\pi}}$
Rectangular-Function	$\mathrm{rect}(\frac{t}{\tau})$	$\lvert\,\tau\,\rvert\,\mathrm{sinc}(\frac{\Omega\,\tau}{2})$
Sinc-Function	$\frac{1}{\pi}\,\mathrm{sinc}(\frac{t}{\tau})$	$\lvert\,\tau\,\rvert\,\mathrm{rect}(\frac{\Omega\,\tau}{2})$
Impulse Train	$\sum_{n=-\infty}^{\infty}\delta(\frac{t}{T}-n)$	$\lvert\,T\,\rvert\sum_{n=-\infty}^{\infty}\delta(\frac{\Omega\,T}{2\pi}-n)$

H-3.9.3 Properties of the Fourier Transform:

Operation	$x(t) = \mathfrak{F}^{-1}\{ X(\Omega) \}$	$X(\Omega) = \mathfrak{F}\{ x(t) \}$
Linearity	$\alpha_1 x_1(t) + \alpha_2 x_2(t)$	$\alpha_1 X_1(\Omega) + \alpha_2 X_2(\Omega)$
Time Shift	$x(t-\tau)$	$X(\Omega)\, e^{-j\,\Omega\,\tau}$
Frequency Shift	$x(t)\, e^{j\,\Omega_0\,t}$	$X(\Omega - \Omega_0)$
Time Reversal	$x(-t)$	$X(-\Omega)$
Conjugation	$x^*(t)$	$X^*(-\Omega)$
Duality	$X(t)$	$2\pi\, x(-\Omega)$
Scaling	$x(\frac{t}{T})$	$\lvert T \rvert \cdot X(\Omega\, T)$
Symmetry	$x(t) \in \mathbb{R}$	$X(\Omega) = X^*(-\Omega)$
Frequency Differentiation	$t\, x(t)$	$j\, \frac{d}{d\Omega}\, X(\Omega)$
Time Differentiation	$\frac{d}{dt}\, x(t)$	$j\, \Omega\, X(\Omega)$
Convolution	$x(t) * h(t)$	$X(\Omega) \cdot H(\Omega)$
Cross-Correlation	$x(t) * y^*(-t)$	$X(\Omega) \cdot Y^*(\Omega)$
Multiplication	$x(t) \cdot y(t)$	$\frac{1}{2\pi}\, [\, X(\Omega) * Y(\Omega)\,]$