Signals & Systems
Handout #4

H-4.1 Elementary Discrete-Domain Functions (Sequences):

Discrete-domain functions are defined for $n \in \mathbb{Z}$.

H-4.1.1 Sequence Notation:

We use the following notation to indicate the elements of a sequence $x[n]$ between index n_L and index n_H:

$$x[n] = \{\, x[n_L],\, x[n_L+1],\, \ldots\,,\, x[n_H-1],\, x[n_H]\,\}.$$

The elements outside of the given range are assumed to be zero (unless stated otherwise). The element that is associated with index $n = 0$ is indicated with an arrow:

$$x[n] = \{\, \ldots\,,\, x[-2],\, x[-1],\, \underset{\uparrow}{x[0]},\, x[1],\, x[2],\, \ldots\}.$$

If the arrow is omitted then the first given element in the sequence is assumed to be the element at index zero $x[n] = \{\, x[0],\, x[1],\, x[2],\, \ldots\,\}$.

H-4.1.2 Step Sequence:

$$\mu[n] = \begin{cases} 0 & \text{for} \quad n < 0 \\ 1 & \text{for} \quad n \geq 0 \end{cases}$$

H-4.1.3 Kronecker Delta Sequence:

$$\delta[n] = \begin{cases} 1 & \text{for} \quad n = 0 \\ 0 & \text{for} \quad n \neq 0 \end{cases}$$

H-4.2 Classification of Discrete-Domain Signals:

We consider discrete-domain signals $x[n]$ that are defined for $n \in \mathbb{Z}$. The range of discrete-domain signals may be real ($x[n] \in \mathbb{R}$) or complex ($x[n] \in \mathbb{C}$).

H-4.2.1 Periodic Signals:

A discrete-domain signal $x[n]$ is *periodic* with *period* N if there is a $N \in \mathbb{Z}$ such that $x[n] = x[n-N]$ for all $n \in \mathbb{Z}$.

H-4.2.2 Symmetric Signals:

A discrete-domain signal $x[n]$ is of *even symmetry* if $x[n] = x[-n]$. It is of *odd symmetry* if $x[n] = -x[-n]$. A (complex-valued) signal is of *even Hermitian symmetry* if $x[n] = x^*[-n]$. It is of *odd Hermitian symmetry* if $x[n] = -x^*[-n]$.

H-4.2.3 Symmetry Decompositions:

A discrete-domain signal $x[n]$ can be decomposed into its

- *even part* $\frac{1}{2}\,(\,x[n] + x[-n]\,)$
- *odd part* $\frac{1}{2}\,(\,x[n] - x[-n]\,)$
- *conjugate symmetric part* $\frac{1}{2}\,(\,x[n]+x^*[-n]\,)$ (even Hermitian symmetry)
- *conjugate antisymmetric part* $\frac{1}{2}\,(\,x[n]-x^*[-n]\,)$ (odd Hermitian symmetry).

H-4.2.4 Bounded Signals:

A discrete-domain signal $x[n]$ is *bounded* if $|\,x[n]\,| \leq \mathcal{B}_x < \infty$ for some finite $\mathcal{B}_x \in \mathbb{R}^+$. (In writing $\mathcal{B}_x$ we imply the smallest number such that $|\,x[n]\,| \leq \mathcal{B}_x$.)

H-4.2.5 Energy Signals:

A discrete-domain signal $x[n]$ is an *energy signal* or *square-summable signal* if its energy $\mathcal{E}_x$ is finite.

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |\,x[n]\,|^2 < \infty$$

H-4.2.6 Power Signals:

A discrete-domain signal $x[n]$ is a *power signal* if its power $\mathcal{P}_x$ is finite.

$$\mathcal{P}_x = \lim_{K\to\infty} \tfrac{1}{2K+1} \sum_{n=-K}^{K} |\,x[n]\,|^2 < \infty$$

A periodic signal $x[n]$ with period N is a power signal with $\mathcal{P}_x = \frac{1}{N}\sum_{n=0}^{N-1} |\,x[n]\,|^2$.

H-4.2.7 Absolutely Summable Signals:

A discrete-domain signal $x[n]$ is *absolutely summable* if

$$\mathcal{S}_x = \sum_{n=-\infty}^{\infty} |\,x[n]\,| < \infty.$$

H-4.2.8 Finite Length Signals:

A discrete-domain signal $x[n]$ is of *finite length* if there exists a n_1 and a n_2 with $n_1 \leq n_2$ such that $x[n] = 0$ for all $n < n_1$ and $n > n_2$. Let $\tilde{n}_1$ denote the largest possible n_1 such that $x[n] = 0$ for all $n < \tilde{n}_1$ and let $\tilde{n}_2$ denote the smallest possible n_2 such that $x[n] = 0$ for all $n > \tilde{n}_2$ then the length of $x[n]$ is defined by:

$$\mathcal{L}_x = \tilde{n}_2 - \tilde{n}_1 + 1.$$

Note that the length of a signal $x[n]$ that is identically equal to zero for all $n \in \mathbb{Z}$ is not defined!

H-4.2.9 Causal and Anti-Causal Signals:

A discrete-domain signal $x[n]$ is *causal* if $x[n] = 0$ for all $n < 0$. It is *anticausal* if $x[n] = 0$ for all $n > 0$.

H-4.3 Elementary Discrete-Domain Signal Operations:

H-4.3.1 Convolution:

The discrete-domain *convolution* of two signals $x[n]$ and $h[n]$ is defined by

$$y[n] = h[n] \circledast x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k].$$

Convolution generally involves folding, shifting, multiplication, and summation.

H-4.3.2 Properties of Convolution:

The discrete-domain *convolution* operator $\circledast$ has the following properties:

a)	Commutativity:	$x[n] \circledast h[n] = h[n] \circledast x[n]$
b)	Distributivity:	$x[n] \circledast (h_1[n] + h_2[n]) = x[n] \circledast h_1[n] + x[n] \circledast h_2[n]$
c)	Associativity:	$(x_1[n] \circledast x_2[n]) \circledast x_3[n] = x_1[n] \circledast (x_2[n] \circledast x_3[n])$
d)	Shift Property:	$h[n] \circledast x[n] = y[n] \;\Rightarrow\; h[n-k] \circledast x[n] = y[n-k]$
e)	Convolution Length:	$y[n] = h[n] \circledast x[n] \;\Rightarrow\; \mathcal{L}_y \leq \mathcal{L}_h + \mathcal{L}_x - 1$

H-4.3.3 Elementary Convolution Identities:

a) $x[n] \circledast \delta[n] = x[n]$ (i.e. $x[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k]$)

b) $\mu[n] \circledast \mu[n] = (n+1)\, \mu[n]$

H-4.3.4 Properties of the Kronecker Delta Sequence:

a)	Sum:	$\sum_{n=-\infty}^{\infty} \delta[n] = 1$
b)	Exchange:	$x[n]\, \delta[n-k] = x[k]\, \delta[n-k]$
c)	Scaling:	$\delta[K\, n] = \delta[n]$ for $K \in \mathbb{Z}$
e)	Convolution:	$x[n] \circledast \delta[n-k] = x[n-k]$
f)	Symmetry:	$\delta[n] = \delta[-n]$

H-4.3.5 Deterministic Correlation:

The *(deterministic) correlation* of two energy signals $x[n]$ and $y[n]$ is defined by

$$r_{xy}[k] = \sum_{n=-\infty}^{\infty} x[n+k]\, y^*[n] = x[k] \circledast y^*[-k].$$

For two power signals $x[n]$ and $y[n]$ we define respectively

$$\tilde{r}_{xy}[k] = \lim_{K \to \infty} \tfrac{1}{2K+1} \sum_{n=-K}^{K} x[n+k]\, y^*[n].$$

For two signals $x[n]$ and $y[n]$ that are both periodic with period N we obtain

$$\tilde{r}_{xy}[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n+k]\, y^*[n].$$

H-4.4 Classification of Discrete-Domain Systems:

We consider discrete-domain systems $\mathfrak{T}$ with input $x[n]$ and output $y[n]$.

$$y[n] = \mathfrak{T}\{\, x[n]\, \}$$

H-4.4.1 Linear Systems:

A discrete-domain system $\mathfrak{T}$ is *linear* if for any two arbitrary input signals $x_1[n]$, $x_2[n]$ and for any two constants α_1, $\alpha_2 \in \mathbb{R}$ (or $\mathbb{C}$) we have

$$\mathfrak{T}\{\, \alpha_1\, x_1[n] + \alpha_2\, x_2[n]\, \} = \alpha_1\, \mathfrak{T}\{\, x_1[n]\, \} + \alpha_2\, \mathfrak{T}\{\, x_2[n]\, \}.$$

H-4.4.2 Time-Invariant Systems:

A discrete-domain system $\mathfrak{T}$ is *time-invariant* if $y[n] = \mathfrak{T}\{\, x[n]\, \}$ implies that $y[n-k] = \mathfrak{T}\{\, x[n-k]\, \}$ for any arbitrary input signal $x[n]$ any arbitrary delay $k \in \mathbb{R}$.

H-4.4.3 Causal Systems:

A discrete-domain system $\mathfrak{T}$ is *causal* if the output $y[n]$ at time n only depends on current and past input values $x[k]$ for $k \leq n$ and/or only depends on past output values $y[k]$ for $k < n$.

H-4.4.4 BIBO Stable Systems:

A discrete-domain system $\mathfrak{T}$ is *bounded-input bounded-output (BIBO) stable* if any bounded input $|\, x[n]\, | \leq \mathcal{B}_x < \infty$ leads to a bounded output $|\, y[n]\, | \leq \mathcal{B}_y < \infty$.

H-4.4.5 Passive and Lossless Systems:

A system with arbitrary *square summable* input $x[n]$ and output $y[n]$ is called *passive* if $\mathcal{E}_y \leq \mathcal{E}_x$. Systems for which $\mathcal{E}_y = \mathcal{E}_x$ for any *square summable* input $x[n]$ are called *lossless*.

H-4.4.6 Up-Sampling and Down-Sampling Systems:

A discrete-domain system that inserts $L-1$ ($L \in \mathbb{N}$) zeros between every element of an input sequence $x[n]$ is called an *up-sampling system* of order L:

$$y[n] = \begin{cases} x[\, n/L\,] & \text{for} \quad n = L\, k \quad \text{with} \quad k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

A discrete-domain system is called a *down-sampling system* of order L if it discards all elements of input $x[n]$ that are not indexed by a multiple of L:

$$y[n] = x[\, n \cdot L\,]$$

H-4.5 Discrete Linear Time-Invariant (DLTI) Systems:

H-4.5.1 Impulse Response:

Let $\mathfrak{T}$ denote a DLTI system. If we let the *impulse response* $h[n]$ of $\mathfrak{T}$ be defined as $h[n] = \mathfrak{T}\{\, \delta[n] \,\}$ then the response of $\mathfrak{T}$ to an arbitrary input $x[n]$ is given by

$$y[n] = x[n] \circledast h[n].$$

H-4.5.2 Causal DLTI Systems:

A DLTI system $\mathfrak{T}$ is *causal* if and only if its *impulse response* $h[n]$ is a causal signal:

$$h[n] = 0 \quad \text{for} \quad n < 0.$$

H-4.5.3 BIBO Stable DLTI Systems:

A DLTI system $\mathfrak{T}$ is *BIBO stable* if and only if its *impulse response* $h[n]$ is absolutely summable, i.e. if $\mathcal{S}_h < \infty$.

H-4.5.4 FIR and IIR Systems:

A DLTI system is called a *finite impulse response system* (FIR system) if the length of the impulse response $h[n]$ is finite, i.e. if $\mathcal{L}_h < \infty$. A DLTI system is called an *infinite impulse response system* (IIR system) if $\mathcal{L}_h = \infty$.

H-4.5.5 Eigenfunctions of DLTI Systems:

Input functions of the form $x[n] = z_0^n$ are *eigenfunctions* of DLTI systems.

$$y[n] = \mathfrak{T}\{\, z_0^n \,\} = h[n] \circledast\, z_0^n = z_0^n \cdot \underbrace{\textstyle\sum_{k=-\infty}^{\infty} h[k]\, z_0^{-k}}_{=H(z_0)} = z_0^n \cdot H(z_0)$$

When passed through a DLTI system, these eigenfunctions remain unchanged up to a constant (possibly complex) gain $H(z_0)$.

H-4.6 The Z-Transform:

H-4.6.1 Definition of the (Bilateral) Z-Transform:

The *(bilateral) z-transform* $X(z)$ of signal $x[n]$ is defined by

$$X(z) = \mathcal{Z}\{\, x[n] \,\} = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$\text{with} \quad \text{ROC:} \;\; 0 \le r_1 < |\, z \,| < r_2 \le +\infty.$$

The *z-transform* always consists of both the complex function $X(z)$ and its associated *region of convergence* (ROC). The region of convergence is the set of all complex values z for which the transform summation converges. The ROC is generally a ring in the complex plane, bounded by an inner radius r_1 and an outer radius r_2 ($r_1, r_2 \in \mathbb{R}^+$). The radius r_1 is determined by the rate of exponential increase/decrease of the causal part of $x[n]$. Similarly, r_2 is determined by the rate of exponential increase/decrease of the anti-causal part of $x[n]$.

H-4.6.2 The Inverse Z-Transform:

The *inverse z-transform* is defined as

$$x[n] = \mathcal{Z}^{-1}\{\, X(z)\,\} = \frac{1}{2\,\pi\,j} \oint_{\mathcal{C}} X(z)\, z^{n-1}\, dz$$

in which the integration contour $\mathcal{C}$ is given by

$$\mathcal{C}\colon r\, e^{j\,\omega} \Big|_{\omega=-\pi}^{\omega=+\pi} \qquad \text{for some fixed} \quad r \in \,]\, r_1, r_2\,[.$$

H-4.6.3 Complex Contour Integration:

If a sufficiently smooth complex contour $\mathcal{C}$ can be described with a parameter description $p(\varphi) \in \mathbb{C}$ for $\varphi \in [\,a, b\,]$ then $\int_{\mathcal{C}} F(s)\, ds = \int_a^b F(p(\varphi))\, p'(\varphi)\, d\varphi$. A complex contour integral can thus be reduced to a conventional Riemann integral.

H-4.6.4 Five Elementary Z-Transform Identities:

$x[n] = \mathcal{Z}^{-1}\{\, X(z)\,\}$	$X(z) = \mathcal{Z}\{\, x[n]\,\}$	ROC:
$\delta[n]$	1	$z \in \mathbb{C}$
$\alpha^n\, \mu[n]$	$\dfrac{z}{z-\alpha}$	$\lvert\, z\,\rvert > \lvert\,\alpha\,\rvert$
$-\alpha^n\, \mu[-n-1]$	$\dfrac{z}{z-\alpha}$	$\lvert\, z\,\rvert < \lvert\,\alpha\,\rvert$
$n\,\alpha^n\, \mu[n]$	$\dfrac{\alpha\, z}{(z-\alpha)^2}$	$\lvert\, z\,\rvert > \lvert\,\alpha\,\rvert$
$-n\,\alpha^n\, \mu[-n-1]$	$\dfrac{\alpha\, z}{(z-\alpha)^2}$	$\lvert\, z\,\rvert < \lvert\,\alpha\,\rvert$

H-4.6.5 The Z-Transform of Causal Signals:

Note that every valid z-transform expression $X(z)$ has only *one* causal inverse transform $x[n]$. We do not need to know the ROC explicitly to find the correct causal inverse of $X(z)$.

H-4.6.6 A Short Table of Z-Transforms of Causal Signals:

$x[n] = \mathcal{Z}^{-1}\{ X(z) \}$	$X(z) = \mathcal{Z}\{ x[n] \}$	ROC:
$\mu[n]$	$\frac{z}{z-1}$	$\lvert z \rvert > 1$
$\alpha^n \cos(\omega_0\, n)\, \mu[n]$	$\frac{z^2 - \alpha z \cos\omega_0}{z^2 - 2\alpha z\cos\omega_0 + \alpha^2}$	$\lvert z \rvert > \lvert \alpha \rvert$
$\alpha^n \sin(\omega_0\, n)\, \mu[n]$	$\frac{\alpha z \sin\omega_0}{z^2 - 2\alpha z\cos\omega_0 + \alpha^2}$	$\lvert z \rvert > \lvert \alpha \rvert$

H-4.6.7 Properties of the Bilateral Z-Transform:

Operation	$x[n] = \mathcal{Z}^{-1}\{ X(z) \}$	$X(z) = \mathcal{Z}\{ x[n] \}$ and ROC[a]
Linearity	$\alpha_1\, x_1[n] + \alpha_2\, x_2[n]$	$\alpha_1\, X_1(z) + \alpha_2\, X_2(z)$ $\quad$ $\text{ROC}_1 \cap \text{ROC}_2$
Time Shift	$x[n-k]$	$X(z)\, z^{-k}$ and same ROC[b]
Modulation	$\alpha^n\, x[n]$	$X(z/\alpha)$ ROC[c] is scaled by $\lvert \alpha \rvert$
Differentiation in Z-Domain	$n\, x[n]$	$-z\, \frac{d}{dz}\, X(z)$ and same ROC
Conjugation	$x^*[n]$	$X^*(z^*)$ and same ROC
Convolution	$x[n] \circledast h[n]$	$X(z) \cdot H(z)$ $\quad$ $\text{ROC}_1 \cap \text{ROC}_2$

[a] The actual ROC of the result of an operation may be larger than the one provided in the table. Check the common literature on z-transforms for the details.

[b] Same ROC possibly except $z = 0$ if $k > 0$.

[c] If the original ROC of $X(z)$ is given by $r_1 < \lvert z \rvert < r_2$ then the scaled ROC of $X(z/\alpha)$ is given by $\lvert \alpha \rvert\, r_1 < \lvert z \rvert < \lvert \alpha \rvert\, r_2$

H-4.7 DLTI Systems and the Z-Transform:

H-4.7.1 Transfer Functions and BIBO Stable Systems:

Let $H(z) = \mathcal{Z}\{\, h[n] \,\}$ denote the z-transform of the impulse response $h[n]$ of a DLTI system. $H(z)$ is called the *transfer function* of the DLTI system. A DLTI system is *BIBO stable* if the *unit circle* ($|\, z \,| = 1$) is contained in the ROC of its transfer function $H(z)$.

H-4.7.2 Linear Constant Coefficient Difference Equations:

Every *linear constant coefficient difference equation* with input $x[n]$ and output $y[n]$ establishes a *causal linear time-invariant system.*

$$\begin{aligned} y[n] = -a_1\, y[n-1] - a_2\, y[n-2] - \ldots \\ \ldots - a_N\, y[n-N] + b_0\, x[n] + b_1\, x[n-1] + b_2\, x[n-2] + \ldots \\ \ldots + b_M\, x[n-M] \end{aligned}$$

By transforming the difference equation into the z-domain we obtain the *transfer function* $H(z)$ of the associated DLTI system. The transfer function of a linear constant coefficient difference equation is rational in variable z:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

Since $H(z)$ is the transfer function of a *causal* system we do not need to explicitly provide its ROC. Furthermore, we can write every rational transfer function of the form above in terms of its poles p_i (for $i = 1 \ldots N$) and zeros z_i (for $i = 1 \ldots M$).

$$H(z) = b_0 \cdot z^{(N-M)} \cdot \frac{(z - z_1)(z - z_2) \ldots (z - z_M)}{(z - p_1)(z - p_2) \ldots (z - p_N)}$$

The term b_0 is often referred to as the *gain* of the system. Note, however, that b_0 is usually *not* equal to the DC gain or the high-frequency gain of a system!

H-4.7.3 Stability of Causal DLTI Systems with Rational Transfer Functions:

A causal DLTI system with a *rational* transfer function $H(z)$ is stable if and only if the magnitude of all of its poles is strictly smaller than one ($|\, p_i \,| < 1$ for $i = 1 \ldots N$), i.e. if all poles are strictly inside of the unit circle.

H-4.7.4 System I/O Description in the Z-Domain:

Due to the convolution theorem of the z-transform we can find the output $y[n]$ of a DLTI system for a given input $x[n]$ conveniently in the Z-Domain:

$$Y(z) = \mathcal{Z}\{\, y[n] \,\} = H(z) \cdot X(z) = \mathcal{Z}\{\, h[n] \,\} \cdot \mathcal{Z}\{\, x[n] \,\}.$$

If $Y(z)$ is rational then we can find its inverse transform $y[n]$ via a partial fraction expansion in z^{-1} and a table lookup.

H-4.8 The Discrete-Time Fourier Transform (DTFT):

H-4.8.1 Definition of the Discrete-Time Fourier Transform:

The *discrete-time Fourier transform* (DTFT) and its inverse are defined by

$$X(\omega) = \text{DTFT}\{\, x[n]\,\} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$\text{and} \qquad x[n] = \text{DTFT}^{-1}\{\, X(\omega)\,\} = \tfrac{1}{2\pi} \int_{2\pi} X(\omega)\, e^{j\omega n}\, d\omega.$$

The existence of the discrete-time Fourier transform is guaranteed for absolutely summable signals. For other signals meaningful definitions for the DTFT may be found, but the existence is not guaranteed in general.

H-4.8.2 Some Elementary DTFT Identities:

$x[n] = \text{DTFT}^{-1}\{\, X(\omega)\,\}$	$X(\omega) = \text{DTFT}\{\, x[n]\,\}$
$x[n] = 1$	$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
$x[n] = \delta[n-k]$	$X(\omega) = e^{-j\omega k}$
$x[n] = e^{j\omega_0 n}$	$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k)$
$x[n] = \mu[n]$	$X(\omega) = \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
$x[n] = \alpha^n\, \mu[n]$ with $\lvert\alpha\rvert < 1$	$X(\omega) = \dfrac{1}{1 - \alpha\, e^{-j\omega}}$
$x[n] = \begin{cases} 1 & \text{for } \lvert n \rvert \le K \\ 0 & \text{for } \lvert n \rvert > K \end{cases}$	$X(\omega) = \dfrac{\sin((K+\frac{1}{2})\,\omega)}{\sin(\frac{\omega}{2})}$
$x[n] = \begin{cases} \omega_0/\pi & \text{for } n = 0 \\ \frac{\sin(\omega_0 n)}{\pi n} & \text{for } n \neq 0 \end{cases}$	$\tilde{X}(\omega) = \begin{cases} 1 & \text{for } \lvert\omega\rvert < \omega_0 \\ 1/2 & \text{for } \lvert\omega\rvert = \omega_0 \\ 0 & \text{for } \lvert\omega\rvert > \omega_0 \end{cases}$ $X(\omega) = \sum_{k=-\infty}^{\infty} \tilde{X}(\omega - 2\pi k)$

Note that we can directly derive the DTFT $X(\omega)$ of a signal $x[n]$ from its z-transform $X(z)$ if the ROC of $X(z)$ contains the unit circle.

$$X(\omega) = X(z)\,|_{\,z=e^{j\omega}} \quad \text{if} \quad e^{j\omega} \in \text{ROC} \quad \text{for} \quad \omega \in [-\pi, \pi]$$

There is an ambiguity in our notation for the z-transform $X(z)$ and the DTFT $X(\omega)$. The distinction is achieved with the name of the independent variable: (z) for the z-transform and (ω) for the DTFT.

H-4.8.3 Properties of the DTFT:

Operation	$x[n] = \text{DTFT}^{-1}\{\, X(\omega)\,\}$	$X(\omega) = \text{DTFT}\{\, x[n]\,\}$
Linearity	$\alpha_1\, x_1[n] + \alpha_2\, x_2[n]$	$\alpha_1\, X_1(\omega) + \alpha_2\, X_2(\omega)$
Time Shift	$x[n-k]$	$X(\omega)\, e^{-j\omega k}$
Frequency Shift	$x[n]\, e^{j\omega_0 n}$	$X(\omega - \omega_0)$
Time Reversal	$x[-n]$	$X(-\omega)$
Conjugation	$x^*[n]$	$X^*(-\omega)$
Frequency Differentiation	$n\, x[n]$	$j\, \frac{d}{d\omega}\, X(\omega)$
Convolution	$x[n] \circledast h[n]$	$X(\omega) \cdot H(\omega)$
Cross-Correlation	$x[n] \circledast y^*[-n]$	$X(\omega) \cdot Y^*(\omega)$
Multiplication	$x[n] \cdot y[n]$	$\frac{1}{2\pi} \int_{2\pi} X(\lambda) Y(\omega - \lambda)\, d\lambda$