Signals & Systems
Signal Theory Handout #5

H-5.1 The Dirac Delta Function:

The Dirac delta function $\delta(t)$ lies at the heart of modern signals and systems theory. The term *function* is technically a misnomer since $\delta(t)$ cannot be defined in the sense of an ordinary function. It represents a member of a broader class of signals called "generalized functions" or "distributions." A rigorous mathematical treatment of distributions is possible, but not part of this handout.

H-5.1.1 Definition of the Dirac Delta Function:

Consider a pulse function $p(t)$ with $\int_{-\infty}^{\infty} p(t)\,dt = 1$ and such that its *weight* is concentrated (in an appropriate mathematical sense) around $t = 0$, i.e. such that

$$x(0) = \lim_{T\to 0} \int_{-\infty}^{\infty} x(t) \cdot \tfrac{1}{T}\, p(\tfrac{t}{T})\,dt \tag{5.1}$$

for any *smooth* and *well behaved* function $x(t)$. We are using the symbol $\delta(t)$ as a notational simplification of equation (5.1):

$$x(0) = \int_{-\infty}^{\infty} x(t) \cdot \delta(t)\,dt. \tag{5.2}$$

There is an infinite number of possible pulse functions $p(t)$ that may be used. We formally write $\delta(t) = \lim\limits_{T\to 0} \frac{1}{T}\, p(\frac{t}{T})$. Examples are:

$$\delta(t) = \lim_{T\to 0} \tfrac{1}{T}\,\mathrm{rect}(\tfrac{t}{T}) \qquad \delta(t) = \lim_{T\to 0} \tfrac{1}{T}\,\Delta(\tfrac{t}{T}) \tag{5.3}$$

$$\delta(t) = \lim_{T\to 0} \tfrac{1}{T}\, e^{-\pi(t/T)^2} \qquad \delta(t) = \lim_{T\to 0} \tfrac{1}{T}\,\mathrm{sinc}(\pi\tfrac{t}{T}) \tag{5.4}$$

Note that for these example pulses we have $\int_{-\infty}^{\infty} \mathrm{rect}(t)\,dt = 1$, $\int_{-\infty}^{\infty} \Delta(t)\,dt = 1$, $\int_{-\infty}^{\infty} e^{-\pi t^2}\,dt = 1$, and $\int_{-\infty}^{\infty} \mathrm{sinc}(\pi t)\,dt = 1$.

H-5.1.2 Properties of the Dirac Delta Function:

a) Area: $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$
b) Sampling: $\int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\,dt = x(\tau)$
c) Exchange: $x(t)\,\delta(t-\tau) = x(\tau)\,\delta(t-\tau)$
d) Scaling: $\delta(\alpha\, t) = \frac{1}{|\alpha|}\,\delta(t)$
e) Convolution: $x(t) * \delta(t-\tau) = x(t-\tau)$
f) Symmetry: $\delta(t) = \delta(-t)$

H-5.1.3 The Fourier Kernel:

The integral $\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\Omega t}\,d\Omega$ does not converge in a conventional sense. We can, however, imbed the integral within the framework of generalized functions:

$$\lim_{T\to 0}\frac{1}{2\pi}\int_{-\pi/T}^{+\pi/T} e^{j\Omega t}\,d\Omega = \lim_{T\to 0}\frac{1}{2\pi}\left[\frac{1}{j\,t}\,e^{j\Omega t}\right]_{-\pi/T}^{+\pi/T} = \lim_{T\to 0}\frac{\frac{1}{2j}\left[\,e^{+j\pi t/T}-e^{-j\pi t/T}\,\right]}{\pi\,t}$$

$$= \lim_{T\to 0}\frac{1}{T}\,\frac{\sin(\pi\frac{t}{T})}{\pi\,\frac{t}{T}} = \lim_{T\to 0}\frac{1}{T}\,\mathrm{sinc}(\pi\tfrac{t}{T}) = \delta(t) \qquad (5.5)$$

$$\Rightarrow\quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\Omega t}\,d\Omega = \delta(t). \qquad (5.6)$$

By substituting $t-\tau$ for t we immediately get:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\Omega(t-\tau)}\,d\Omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\Omega t}\cdot e^{-j\Omega\tau}\,d\Omega = \delta(t-\tau). \qquad (5.7)$$

H-5.1.4 Linear Time-Invariant Systems:

Let $\mathfrak{T}$ denote a continuous-domain *linear* and *time-invariant* (LTI) system. With input $x(t)=\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$ we can write output $y(t)$ as:

$$y(t) = \mathfrak{T}\{\,x(t)\,\} = \mathfrak{T}\{\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau\,\} = \int_{-\infty}^{+\infty} x(\tau)\cdot\mathfrak{T}\{\,\delta(t-\tau)\,\}\,d\tau \qquad (5.8)$$

Furthermore, by defining the impulse response $h(t)=\mathfrak{T}\{\,\delta(t)\,\}$, and by using the fact that the system is time-invariant, i.e. $h(t-\tau)=\mathfrak{T}\{\,\delta(t-\tau)\,\}$, we have:

$$y(t) = \mathfrak{T}\{\,x(t)\,\} = \int_{-\infty}^{+\infty} x(\tau)\cdot h(t-\tau)\,d\tau = x(t)*h(t). \qquad (5.9)$$

The *convolution operator* $*$ can therefore be used in conjunction with the impulse response $h(t)$ to fully describe the behavior of a continuous-domain LTI system.

H-5.2 The Fourier Transform:

H-5.2.1 Definition of the Fourier Transform:

The *continuous time Fourier transform* and its *inverse transform* are defined by

$$X(\Omega) = \mathfrak{F}\{\,x(t)\,\} = \int_{-\infty}^{\infty} x(t)\,e^{-j\Omega t}\,dt \qquad (5.10)$$

$$\text{and}\qquad x(t) = \mathfrak{F}^{-1}\{\,X(\Omega)\,\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\,e^{j\Omega t}\,d\Omega \qquad (5.11)$$

The existence of the Fourier transform is guaranteed for absolutely integrable signals. For other signals meaningful definitions for Fourier transforms may be found, but the existence is not guaranteed in general.

H-5.2.2 The Invertibility of the Fourier Transform:

With our notion of the Dirac delta it is straightforward to show that the inverse Fourier transform $\mathfrak{F}^{-1}\{\, X(\Omega)\, \}$ returns the original input signal $x(t)$:

$$\begin{aligned}
\mathfrak{F}^{-1}\{\, X(\Omega)\, \} &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\, e^{j\Omega t}\, d\Omega && (5.12)\\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)\, e^{-j\Omega\tau}\, d\tau\right) e^{j\Omega t}\, d\Omega && (5.13)\\
&= \int_{-\infty}^{\infty} x(\tau)\left(\tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\Omega(t-\tau)}\, d\Omega\right) d\tau && (5.14)\\
&= \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau = x(t). && (5.15)
\end{aligned}$$

H-5.2.3 Elementary Properties of the Fourier Transform:

Let $x(t)$ and $X(\Omega)$ and $h(t)$ and $H(\Omega)$ denote respective Fourier transform pairs, i.e. $X(\Omega) = \mathfrak{F}\{\, x(t)\, \}$ and $H(\Omega) = \mathfrak{F}\{\, h(t)\, \}$, then:

a) Time Shift: $\mathfrak{F}\{\, x(t-\tau)\, \} = X(\Omega)\cdot e^{-j\Omega\tau}$

b) Frequency Shift: $\mathfrak{F}\{\, x(t)\cdot e^{j\Omega_0 t}\, \} = X(\Omega-\Omega_0)$

c) Convolution: $\mathfrak{F}\{\, x(t) * h(t)\, \} = X(\Omega)\cdot H(\Omega)$

d) Multiplication: $\mathfrak{F}\{\, x(t)\cdot h(t)\, \} = \frac{1}{2\pi}\,[\, X(\Omega) * H(\Omega)\,]$

H-5.3 Geometric Sums and Series:

H-5.3.1 The Geometric Sum and Series:

A geometric sum is a sum of terms of the form z^k with $z \in \mathbb{C}$ and $k \in \mathbb{Z}$. For an arbitrary complex number $z \neq 1$ we have:

$$\begin{aligned}
(1-z)\cdot\sum_{k=0}^{N-1} z^k = \sum_{k=0}^{N-1}(z^k - z^{k+1}) &= 1 + \left(\sum_{k=1}^{N-1} z^k\right) - \left(\sum_{k=0}^{N-2} z^{k+1}\right) - z^{(N-1)+1}\\
&= 1 + \left(\sum_{k=1}^{N-1} z^k\right) - \left(\sum_{k=1}^{N-1} z^k\right) - z^N = 1 - z^N. \qquad (5.16)
\end{aligned}$$

We can find a closed form expression for the geometric sum $\sum_{k=0}^{N-1} z^k$ by dividing both sides of equation (5.16) by $(1-z)$. Furthermore, for $z = 1$ we have $\sum_{k=0}^{N-1} z^k = N$. Combined with equation (5.16) we get:

$$\sum_{k=0}^{N-1} z^k = \begin{cases} \frac{1-z^N}{1-z} & \text{for } z \neq 1 \\ N & \text{for } z = 1. \end{cases} \qquad (5.17)$$

For any complex number z such that $|z| < 1$ we have $\lim_{N\to\infty} z^N = 0$ and therefore:

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \quad \text{for} \quad |z| < 1. \qquad (5.18)$$

H-5.3.2 A Modified Geometric Sum:

Equation (5.17) can be modified to consider a summation from $k = -M+1$ to $+M-1$ instead of a summation from $k = 0$ to N:

$$\sum_{k=-M+1}^{M-1} z^k = -z^0 + \sum_{k=-M+1}^{0} z^k + \sum_{k=0}^{M-1} z^k = -1 + \sum_{k=0}^{M-1} z^{-k} + \sum_{k=0}^{M-1} z^k. \quad (5.19)$$

Note that $z^{-k} = (z^{-1})^k$. We can, therefore, use equation (5.17) twice, once with argument z^{-1} and once with argument z. For $z \neq 1$ we obtain:

$$\begin{aligned}\sum_{k=-M+1}^{M-1} z^k &= -1 + \frac{z}{z} \cdot \frac{1 - z^{-M}}{1 - z^{-1}} + \frac{1 - z^M}{1 - z} = -\frac{z-1}{z-1} + \frac{z - z^{-M+1}}{z-1} + \frac{z^M - 1}{z-1} \\ &= \frac{-z + 1 + z - z^{-M+1} + z^M - 1}{z-1} = \frac{z^{-1/2}}{z^{-1/2}} \cdot \frac{z^M - z^{-M+1}}{z-1} \\ &= \frac{z^{+M-1/2} - z^{-M+1/2}}{z^{+1/2} - z^{-1/2}}. \end{aligned} \quad (5.20)$$

If we include the result $\sum_{k=-M+1}^{M-1} z^k = 2M-1$ for $z = 1$ then we get:

$$\sum_{k=-M+1}^{M-1} z^k = \begin{cases} \frac{z^{+M-1/2} - z^{-M+1/2}}{z^{+1/2} - z^{-1/2}} & \text{for } z \neq 1 \\ 2M-1 & \text{for } z = 1. \end{cases} \quad (5.21)$$

H-5.3.3 The Generation of an Impulse Train:

The substitution of $z = e^{j\omega}$ in equation (5.21) leads to the term:

$$\left. \frac{z^{+M-1/2} - z^{-M+1/2}}{z^{+1/2} - z^{-1/2}} \right|_{z=e^{j\omega}} = \frac{\frac{1}{2j}\left(e^{j\omega(M-1/2)} - e^{-j\omega(M-1/2)}\right)}{\frac{1}{2j}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} = \frac{\sin(\omega(M-1/2))}{\sin(\omega/2)}.$$

Note also that $z = e^{j\omega} = 1$ for $\omega = 2\pi k$ (with $k \in \mathbb{Z}$) and therefore we define:

$$\Phi_M(\omega) = \sum_{k=-M+1}^{M-1} e^{j\omega k} = \begin{cases} \frac{\sin(\omega(M-1/2))}{\sin(\omega/2)} & \text{for } \omega \neq 2\pi k \text{ with } k \in \mathbb{Z} \\ 2M-1 & \text{for } \omega = 2\pi k \text{ with } k \in \mathbb{Z}. \end{cases} \quad (5.22)$$

The limit $M \to \infty$ in equation (5.22) does not exist in a conventional sense since $|e^{j\omega}| = 1$ for $\omega \in \mathbb{R}$. We can, however, imbed the limiting process within the framework of generalized functions. Note that $\int_{-\pi}^{\pi} \Phi_M(\omega)\, d\omega = 2\pi$ independent of M. Furthermore, $\Phi_M(\omega)$ is 2π periodic, i.e. $\Phi_M(\omega) = \Phi_M(\omega \pm 2\pi)$. Lastly, with increasing M the area underneath function $\Phi_M(\omega)$ becomes more and more concentrated around multiples of 2π. One can formally show that:

$$\lim_{M\to\infty} \tfrac{1}{2\pi}\, \mathrm{rect}(\tfrac{\omega}{2\pi}) \cdot \Phi_M(\omega) = \delta(\omega). \quad (5.23)$$

Since $\Phi_M(\omega)$ is 2π periodic we can argue that:

$$\tfrac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{j\omega k} = \lim_{M\to\infty} \tfrac{1}{2\pi}\,\Phi_M(\omega) = \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k). \tag{5.24}$$

By substituting $k=-n$ we have $\sum_{k=-\infty}^{\infty} e^{j\omega k} = \sum_{n=-\infty}^{\infty} e^{-j\omega n}$ and therefore:

$$\sum_{k=-\infty}^{\infty} e^{\pm j\omega k} = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k). \tag{5.25}$$

An alternative form of equation (5.25) can be derived by noting that $\delta(\omega - 2\pi k) = \delta(2\pi(\frac{\omega}{2\pi} - k)) = \frac{1}{2\pi}\delta(\frac{\omega}{2\pi} - k)$. We can replace ω with $2\pi t/T$ to obtain:

$$\sum_{k=-\infty}^{\infty} e^{\pm j2\pi tk/T} = \sum_{k=-\infty}^{\infty} \delta(\tfrac{t}{T} - k) = T\sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{5.26}$$

The generalized function $\sum_{k=-\infty}^{\infty} \delta(t-kT)$ implied in equation (5.26) is frequently referred to as an *impulse train.*

H-5.4 The Complex Fourier Series:

The (complex) *Fourier series expansion* is defined for continuous-domain periodic signals with period $\tau > 0$, i.e. $x(t) = x(t-\tau)$:

$$x(t) = \mathrm{FS}^{-1}\{\, X_k \,\} = \sum_{k=-\infty}^{\infty} X_k\, e^{j2\pi kt/\tau} \tag{5.27}$$

$$\text{with} \quad X_k = \mathrm{FS}\{\, x(t) \,\} = \tfrac{1}{\tau}\int_{\tau} x(t)\, e^{-j2\pi kt/\tau}\, dt. \tag{5.28}$$

The notation $(\int_{\tau} \ldots\, dt)$ indicates integration over an arbitrary time interval of length τ, i.e. $(\int_{\tau} \ldots\, dt) = (\int_{t_0}^{t_0+\tau} \ldots\, dt)$ for any arbitrary $t_0 \in \mathbb{R}$.

H-5.5 Properties of Impulse Trains:

H-5.5.1 Definition of the Shah-Function:

An *impulse train* is oftentimes compactly written via the *shah-function:*

$$\underline{\mathrm{III}}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n). \tag{5.29}$$

H-5.5.2 Elementary Properties of the Shah-Function:

$$\text{Scaling:} \quad \tfrac{1}{|T|}\,\underline{\mathrm{III}}(\tfrac{t}{T}) \;=\; \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{5.30}$$

$$\text{Periodicity:} \quad \underline{\mathrm{III}}(\tfrac{t-T}{T}) \;=\; \underline{\mathrm{III}}(\tfrac{t}{T}) \tag{5.31}$$

H-5.5.3 The Fourier Series Expansion of the Shah-Function:

With $x(t) = \underline{\mathrm{III}}(\frac{t}{T})$ and $T > 0$ we have $x(t) = x(t-T)$. Therefore, $x(t)$ is periodic with period T and we can expand $x(t)$ according to H-5.4:

$$\begin{aligned} X_k = \mathrm{FS}\{\, x(t) \,\} &= \tfrac{1}{T} \int_{-T/2}^{+T/2} [\; T \sum_{n=-\infty}^{\infty} \delta(t-nT) \;]\, e^{-j2\pi kt/T}\, dt \qquad (5.32) \\ &= \sum_{n=-\infty}^{\infty} \int_{-T/2}^{+T/2} \delta(t-nT)\, \underbrace{e^{-j2\pi knT/T}}_{=1}\, dt = \sum_{n=-\infty}^{\infty} \underbrace{\int_{-T/2}^{+T/2} \delta(t-nT)\, dt}_{=\delta[n]} = 1. \end{aligned}$$

The Fourier series expansion of the impulse train $x(t) = \underline{\mathrm{III}}(\frac{t}{T})$ is thus $X_k = 1$ for all $k \in \mathbb{Z}$. A resubstitution of this result into the Fourier series resynthesis formula from H-5.4 reconfirms our result (5.26) from section H-5.3.3:

$$x(t) = \mathrm{FS}^{-1}\{\, X_k \,\} \quad \Rightarrow \quad T \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{k=-\infty}^{\infty} e^{j2\pi kt/T}. \qquad (5.33)$$

H-5.5.4 The Fourier Transform of the Shah-Function:

Result (5.25) allows us to readily compute the Fourier transform of an impulse train. For $T > 0$ we have:

$$\begin{aligned} \mathfrak{F}\{\, \tfrac{1}{T}\, \underline{\mathrm{III}}(\tfrac{t}{T}) \,\} &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t-nT)\; e^{-j\,\Omega\, t}\, dt \qquad (5.34) \\ &= \sum_{n=-\infty}^{\infty} e^{-j\,\Omega\, T\, n} \underbrace{\int_{-\infty}^{\infty} \delta(t-nT)\, dt}_{=1} = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega T - 2\pi k) \\ &= \sum_{k=-\infty}^{\infty} \delta(\tfrac{\Omega T}{2\pi} - k) = \underline{\mathrm{III}}(\tfrac{\Omega T}{2\pi}). \qquad (5.35) \end{aligned}$$

The important result $\mathfrak{F}\{\, \frac{1}{T}\, \underline{\mathrm{III}}(\frac{t}{T}) \,\} = \underline{\mathrm{III}}(\frac{\Omega T}{2\pi})$ states that the Fourier transform of an impulse train (in time) is an impulse train (in frequency).

H-5.6 Modelling of Discrete-Domain Signals:

H-5.6.1 The Sampled Signal:

When we sample a continuous-domain signal $x(t)$ with sampling time $T > 0$ then we obtain the discrete-domain sequence $x[n] = x(nT)$. One can model the sampling process in continuous time with the associated *sampled signal* $x_d(t)$:

$$x_d(t) = x(t) \cdot \tfrac{1}{T}\, \underline{\mathrm{III}}(\tfrac{t}{T}) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} x[n] \cdot \delta(t-nT). \qquad (5.36)$$

H-5.6.2 The Fourier Transform of Sampled Signals (Part I):

The following computation explores the connection between the Fourier transform of the sampled signal $X_d(\Omega)$ and the Fourier transform $X(\Omega)$ of the underlying continuous-domain signal:

$$X_d(\Omega) = \mathfrak{F}\{\, x_d(t) \,\} = \mathfrak{F}\{\, x(t) \cdot \tfrac{1}{T}\,\underline{\mathrm{III}}(\tfrac{t}{T}) \,\} = \tfrac{1}{2\pi}\,[\, X(\Omega) * \mathfrak{F}\{\, \tfrac{1}{T}\,\underline{\mathrm{III}}(\tfrac{t}{T}) \,\} \,] \quad (5.37)$$

$$= \tfrac{1}{2\pi}\,[\, X(\Omega) * \underline{\mathrm{III}}(\tfrac{\Omega T}{2\pi}) \,] = \tfrac{1}{2\pi}\,[\, \sum_{k=-\infty}^{\infty} X(\Omega) * \delta(\tfrac{\Omega T}{2\pi} - k) \,] \quad (5.38)$$

$$= \tfrac{1}{T}\,[\, \sum_{k=-\infty}^{\infty} X(\Omega) * \delta(\Omega - \tfrac{2\pi k}{T}) \,] = \tfrac{1}{T} \sum_{k=-\infty}^{\infty} X(\Omega - \tfrac{2\pi k}{T}). \quad (5.39)$$

As a result, $X_d(\Omega)$ is equal to a (scaled) periodic repetition of $X(\Omega)$ with period $\frac{2\pi}{T}$. In short: *Sampling in time is equivalent to periodic repetition in frequency.*

H-5.6.3 The Fourier Transform of Sampled Signals (Part II):

Now we are turning our attention to the connection between the Fourier transform of the sampled signal $X_d(\Omega)$ and an appropriate notion of Fourier transform for the associated discrete-time signal $x[n]$:

$$X_d(\Omega) = \mathfrak{F}\{ \sum_{n=-\infty}^{\infty} x[n] \cdot \delta(t - nT) \,\} \quad (5.40)$$

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot \mathfrak{F}\{\, \delta(t - nT) \,\} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\,\Omega\, T n}. \quad (5.41)$$

For convenience we are introducing a notation for *normalized frequency*:

$$\omega = \Omega T. \quad (5.42)$$

Furthermore, motivated by result (5.41) we define the following notion of a *discrete-time Fourier transform* (DTFT) for discrete-time signals $x[n]$:

$$X(\omega) = \mathrm{DTFT}\{\, x[n] \,\} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \quad (5.43)$$

(one can show that $\quad x[n] = \mathrm{DTFT}^{-1}\{\, X(\omega) \,\} = \frac{1}{2\pi} \int_{2\pi} X(\omega)\, e^{j\omega n}\, d\omega.$)

The existence of the discrete-time Fourier transform is guaranteed for absolutely summable signals. For other signals meaningful definitions for the DTFT may be found, but the existence is not guaranteed in general. With the DTFT we can conveniently rewrite result (5.41) as:

$$X_d(\Omega) = X(\omega)|_{\omega = \Omega T} \quad \text{with} \quad X(\omega) = \mathrm{DTFT}\{\, x[n] \,\}. \quad (5.44)$$

The *discrete-time Fourier transform* (DTFT) is therefore a natural extension of the continuous time Fourier transform.

H-5.6.4 Signal Reconstruction in the Frequency Domain:

If $X(\Omega)$ is bandlimited such that $X(\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$ then:

$$T\,\mathrm{rect}(\tfrac{\Omega T}{2\pi}) \cdot X_d(\Omega) = \sum_{k=-\infty}^{\infty} \underbrace{\mathrm{rect}(\tfrac{\Omega T}{2\pi}) \cdot X(\Omega - \tfrac{2\pi k}{T})}_{=0 \text{ if } k \neq 0} = X(\Omega). \tag{5.45}$$

It is thus possible to reconstruct the spectrum $X(\Omega)$ of a continuous-time signal $x(t)$ from the spectrum of the associated sampled signal $x_d(t)$ if $X(\Omega)$ is appropriately bandlimited.

H-5.6.5 Signal Reconstruction in the Time Domain:

If $X(\Omega)$ is bandlimited such that $X(\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$ then we can take the inverse Fourier transform of result (5.45):

$$\mathfrak{F}^{-1}\{\, X(\Omega)\,\} = \mathfrak{F}^{-1}\{\, T\,\mathrm{rect}(\tfrac{\Omega T}{2\pi}) \cdot X_d(\Omega)\,\} \tag{5.46}$$

$$\Rightarrow \qquad x(t) = \mathfrak{F}^{-1}\{\, T\,\mathrm{rect}(\tfrac{\Omega T}{2\pi})\,\} * x_d(t) \tag{5.47}$$

$$= \mathrm{sinc}(\tfrac{\pi t}{T}) \; * \sum_{n=-\infty}^{\infty} x[n]\, \delta(t - nT) \tag{5.48}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot [\, \mathrm{sinc}(\tfrac{\pi t}{T}) * \delta(t - nT)\,] \tag{5.49}$$

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\; \mathrm{sinc}(\pi\,(\tfrac{t}{T} - n)). \tag{5.50}$$

A signal $x(t)$ that is bandlimited (in frequency) between $-\frac{\pi}{T} < \Omega < \frac{\pi}{T}$ can therefore be *perfectly* reconstructed from its samples $x[n] = x(nT)$ (*Sampling Theorem*).

H-5.7 The Fourier Transform of Periodic Signals:

Consider a continuous-domain periodic signals with period $\tau > 0$, i.e. $x(t) = x(t - \tau)$. The signal $x(t)$ can be written as the infinite repetition of one of its periods:

$$[\,\mathrm{rect}(\tfrac{t}{\tau}) \cdot x(t)\,] * \sum_{k=-\infty}^{\infty} \delta(t - k\tau) = \sum_{k=-\infty}^{\infty} [\,\mathrm{rect}(\tfrac{t}{\tau}) \cdot x(t)\,] * \delta(t - k\tau) \tag{5.51}$$

$$= \sum_{k=-\infty}^{\infty} \mathrm{rect}(\tfrac{t-k\tau}{\tau}) \cdot \underbrace{x(t - k\tau)}_{=x(t)} \tag{5.52}$$

$$= x(t) \cdot \underbrace{\sum_{k=-\infty}^{\infty} \mathrm{rect}(\tfrac{t}{\tau} - k)}_{=1} = x(t). \tag{5.53}$$

Taking the Fourier transform on both sides yields:

$$X(\Omega) = \mathfrak{F}\{\, x(t)\,\} = \mathfrak{F}\{\, [\,\mathrm{rect}(\tfrac{t}{\tau}) \cdot x(t)\,] * \sum_{k=-\infty}^{\infty} \delta(t - k\tau)\,\} \tag{5.54}$$

$$= \mathfrak{F}\{\, \mathrm{rect}(\tfrac{t}{\tau}) \cdot x(t)\,\} \cdot \mathfrak{F}\{\, \tfrac{1}{\tau}\, \underline{\mathrm{III}}(\tfrac{t}{\tau})\,\} \tag{5.55}$$

$$= \left[\int_{-\infty}^{\infty} \mathrm{rect}(\tfrac{t}{\tau}) \cdot x(t)\, e^{-j\Omega t}\, dt \right] \cdot \tfrac{2\pi}{\tau} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\tfrac{2\pi}{\tau}) \tag{5.56}$$

$$= 2\pi \sum_{k=-\infty}^{\infty} \left[\tfrac{1}{\tau} \int_{-\tau/2}^{\tau/2} x(t)\, e^{-j\Omega t}\, dt \right] \cdot\ \delta(\Omega - k\tfrac{2\pi}{\tau}) \tag{5.57}$$

$$= 2\pi \sum_{k=-\infty}^{\infty} \underbrace{\left[\tfrac{1}{\tau} \int_{-\tau/2}^{\tau/2} x(t)\, e^{-j2\pi kt/\tau}\, dt \right]}_{=X_k} \cdot\ \delta(\Omega - k\tfrac{2\pi}{\tau}) \tag{5.58}$$

$$\Rightarrow \quad X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} X_k\ \delta(\Omega - k\tfrac{2\pi}{\tau}). \tag{5.59}$$

The Fourier transform of periodic signals is *discrete* in frequency. The weights of the spectral impulses are determined by the complex Fourier series coefficients X_k.

H-5.8 The Fourier Transform of Discrete Periodic Signals:

Assume that a discrete-time periodic signal $x[n] = x[n-N]$ with period $N > 0$ is obtained from sampling a continuous-time periodic signal $x(t) = x(t - NT)$ with period NT and sampling time $T > 0$, i.e. $x[n] = x(nT)$.

$$x_p(t) = x(t) \cdot \tfrac{1}{T}\, \underline{\mathrm{III}}(\tfrac{t}{T}) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x[n] \cdot \delta(t - nT). \tag{5.60}$$

Since $x(t)$ is periodic we can write its Fourier transform after H-5.7 as:

$$X_p(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} X_k\ \delta(\Omega - k\tfrac{2\pi}{NT}) \tag{5.61}$$

$$\text{with} \qquad X_k = \tfrac{1}{NT} \int_{NT} x_p(t)\, e^{-j2\pi kt/NT}\, dt \tag{5.62}$$

$$= \tfrac{1}{NT} \int_{NT} [\, \sum_{n=-\infty}^{\infty} x[n] \cdot \delta(t - nT)\ ]\, e^{-j2\pi kt/NT}\, dt \tag{5.63}$$

$$= \tfrac{1}{NT} \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi knT/NT} \underbrace{\int_{NT} \delta(t - nT)\, dt}_{=\,\mu[n]-\mu[n-N]} \tag{5.64}$$

$$= \tfrac{1}{NT} \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N}. \tag{5.65}$$

Motivated, by equation (5.61) we define the notion of a *discrete Fourier transform* (DFT) for a discrete-time periodic signal $x[n] = x[n-N]$ as:

$$X[k] = \mathrm{DFT}_N\{\, x[n]\,\} = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N} \tag{5.66}$$

$$\text{(one can also show that} \quad x[n] = \mathrm{DFT}_N^{-1}\{\, X[k]\,\} = \tfrac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j2\pi kn/N}.) \tag{5.67}$$

Using the definition of the DFT we can write:

$$X_p(\Omega) = \tfrac{2\pi}{NT}\sum_{k=-\infty}^{\infty} X[k]\ \delta(\Omega - k\tfrac{2\pi}{NT}).$$

The *discrete Fourier transform* (DFT) is the natural extension of the continuous-time Fourier transform for signals that are both discrete *and* periodic. Note that both $x[n]$ and $X[k]$ are discrete and periodic with period N:

$$X[k-N] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi(k-N)\,n/N} = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N}\ \underbrace{e^{j2\pi Nn/N}}_{=1} = X[k]. \tag{5.68}$$

The properties of the DFT are very similar to the ones of the continuous-time Fourier transform and the DTFT, but in a *cyclic* sense. For example, we can define the notion of a *cyclic convolution* of two periodic sequences $x[n] = x[n-N]$ and $h[n] = h[n-N]$ via:

$$y[n] = x[n] \circledN h[n] = \sum_{k=0}^{N-1} x[k]\cdot h[n-k] = \mathrm{DFT}_N^{-1}\{\, X[k]\cdot H[k]\,\}. \tag{5.69}$$

Since, both $x[n]$ and $X[k]$ are periodic with period N it is usually sufficient to consider/compute them for the range $n = 0 \ldots N-1$ and $k = 0 \ldots N-1$ only. As such, it is possible to use the DFT formula as a means to calculate samples of the DTFT of certain time-limited signals:

$$\text{If } \ x[n] = 0 \ \text{ for } \ n < 0 \ \text{ and } \ n \geq N \ \text{ then } \ X[k] = X(\omega)\big|_{\omega=\frac{2\pi k}{N}}. \tag{5.70}$$

Furthermore, the independent variable k of the DFT is discrete and not continuous. We can therefore use computers to evaluate the DFT formula numerically for arbitrary (time-limited) input signals. Fast algorithms for the computation of DFTs have become collectively known as *fast Fourier transforms* (FFTs) and are readily available in computational software packages such as MATLAB and LabView.

A technique that is frequently used in the context of N-point DFTs is *zero padding.* Consider two non-periodic sequences $x[n]$ and $h[n]$ such that $\mathcal{L}_x + \mathcal{L}_h \leq N$ and

$$\tilde{x}[n] = \tilde{x}[n-N] \text{ with } \tilde{x}[n] = \{\ldots, \underbrace{x[0], x[1], \ldots, x[\mathcal{L}_x - 1], 0, 0, \ldots, 0}_{N \text{ points}}, \ldots\}. \tag{5.71}$$

We assume a time alignment such that $\tilde{x}[0] = x[0]$. Let $\tilde{h}[n]$ be defined analogously. One can then show that for $n = 0 \ldots N-1$ we have $x[n] \circledast h[n] = \tilde{x}[n] \circledN \tilde{h}[n]$. A fast computation of this convolution operation can be accomplished via FFTs.