1 The Compton Effect and Thermal Broadening

In class a question was raised about possible shifts in the wavelength of scattered X-rays due to the random thermal motion of electrons from which they may scatter. The Doppler effect predicts that, when a wave of wavelength $\lambda$ emitted by a stationary source is scattered from a target which moves with speed $u$, the wavelength of the scattered wave $\lambda'$ has range

$$\lambda \leq \lambda' \leq \frac{c + u}{c - u} \lambda$$

where $c$ is the speed of light.

One possible source of wavelength shifts is the thermal motion of electrons. If these are at temperature $T$, then their average kinetic energy is $3kT/2$ where $k = 1.381 \times 10^{-23}$ J/K is Boltzmann's constant.

a) Denote the wavelength of the incident X-rays by $\lambda_i$ and find an expression for the maximum shift in wavelength due to thermal Doppler effects, $\Delta \lambda_{Dop}$, in terms of $c, T, \lambda_i$ and the mass of the electron $m_e = 9.11 \times 10^{-31}$ kg.

b) Denote the maximum shift due to the Compton effect by $\Delta \lambda_{Comp}$. Determine an expression for $\Delta \lambda_{Dop}/\Delta \lambda_{Comp}$ in terms of the parameters in the previous problem and the Compton wavelength for the electron.

c) Suppose that X-rays of wavelength 0.0711 nm are incident on electrons in graphite at temperature 300 K. Calculate $\Delta \lambda_{Dop}/\Delta \lambda_{Comp}$. Are the thermal Doppler effects important in this case?

2 Bohr quantization and the three dimensional harmonic oscillator

Consider an object of mass $m$ and which is free to move in three dimensions. The object is connected by a spring to a fixed point. Using a coordinate system in which the fixed end of the spring is at the origin the force exerted by the spring on the object is

$$\vec{F} = -m\omega^2 r \hat{r}$$

where $r$ is the distance from the origin to the object, $\hat{r}$ is the unit vector directed from the origin to object, and $\omega$ is the frequency of oscillation. The potential energy of this oscillator is

$$U = \frac{1}{2} m\omega^2 r^2.$$ 

The purpose of this exercise is to emulate the Bohr quantization scheme for the oscillator and to determine its energy levels (the answer is remarkably close to the full quantum mechanical treatment).
a) Newton mechanics shows that the oscillator can undergo orbital motion with constant velocity. Use Newton’s second law to determine an expression for the velocity in terms of the radius of orbit, \( r \), and \( m \) and \( \omega \).

b) Determine an expression for the total mechanical energy and the angular momentum of the object in terms of \( r, m \) and \( \omega \).

c) Apply the Bohr quantization rules for angular momentum and determine the allowed radii of orbits and the associated energies. Determine the difference in energies between adjacent energy levels. What are the possible frequencies of light which are emitted?

d) In general the force and potential in these isotropic cases (same in all directions) are related via

\[
\vec{F} = -\frac{\partial U}{\partial r} \hat{r}.
\]

Suppose that

\[
U(r) = kr^\alpha
\]

where \( k \) and \( \alpha \) are constants with the same signs. Repeat the above analysis for this case, determining the allowed energy levels and radii.