1) For energy $E = hf = kT$ 
According to Planck, 
$$ \bar{E} = \int_{0}^{\infty} f(E) \cdot E \, dE = \int_{A}^{\infty} E^{-\frac{E}{kT}} \cdot E \, dE $$ 

The fraction of oscillators with energy $E$ 

$$ \therefore \bar{E} = \frac{E}{e^{E/kT} - 1} \quad \text{where} \quad E = hf $$

\[ \therefore \text{for } E = kT, \]

$$ \bar{E} = \frac{1}{e^{kT} - 1} (kT) = 0.58 \, (kT) $$

(ii) for $E = hf = 10kT$

$$ \Rightarrow \bar{E} = \frac{1}{e^{10kT} - 1} (10 \, kT) = 4.5 \times 10^{-4} \, kT $$

(iii) for $E = hf = 1/10kT$

$$ \Rightarrow \bar{E} = \frac{1}{e^{10kT} - 1} (1/10 \, kT) = 0.95 \, kT $$

To summarize:

<table>
<thead>
<tr>
<th>Energy (kT)</th>
<th>Average Energy (kT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.58</td>
</tr>
<tr>
<td>10</td>
<td>4.5 \times 10^{-4}</td>
</tr>
<tr>
<td>1/10</td>
<td>0.95</td>
</tr>
</tbody>
</table>

(high frequency oscillators have a much lower average energy due to the weighted distribution weighted to low energy end).
Stefan-Boltzmann Law.

Planck's energy density distribution function: \( u(\lambda) = \frac{8\pi\alpha}{\lambda^5} \frac{e^{hc/\lambda kT} - 1}{e^{hc/\lambda kT} - 1} \)

a) \( u(\lambda) \) = energy per unit volume in the cavity contributed by the wavelength, \( \lambda \).

b) To determine the total energy density, \( U \).

\[
U = \int_0^\infty u(\lambda) d\lambda = \int_0^\infty 8\pi\alpha \frac{\lambda^{-5}}{(e^{hc/\lambda kT} - 1)} d\lambda
\]

Let \( x = \frac{hc}{\lambda kT} \):

\[
U = 8\pi\alpha \frac{h/\lambda kT}{(e^{hc/\lambda kT} - 1)} \int_0^\infty x^5 \left( \frac{e^{hc/\lambda kT} - 1}{hc/\lambda kT} \right) \frac{1}{x^2} dx
\]

\[
\lambda = \frac{hc}{kT} x^{-1}
\]

\[
\Rightarrow d\lambda = -\frac{hc}{kT} x^{-2} dx
\]

\[
\therefore U = 8\pi\alpha \left( \frac{kT}{hc} \right)^4 \int_0^\infty x^3 \frac{1}{e^{x} - 1} dx
\]

\[
= -8\pi\alpha \left( \frac{kT}{hc} \right)^4 \int_0^\infty \frac{x^3}{e^x - 1} dx
\]

\[
\Rightarrow U = 8\pi\alpha \left( \frac{kT}{hc} \right)^4 \int_0^\infty \frac{x^3}{e^x - 1} dx , \text{ as required}
\]

c) Now,

\[
\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}
\]

\[
\therefore U = 8\pi\alpha \left( \frac{kT}{hc} \right)^4 \cdot \frac{\pi^4}{15} = \left[ \frac{8\pi^5}{15} \frac{k^4}{(hc)^{10}} \right] T^4
\]

Stefan-Boltzmann \( \Rightarrow R = \sigma T^4 \)

where \( R \) = power radiated per unit area
From classical electromagnetism, $R = \frac{1}{4} CV$

\[ R = \frac{1}{4} CV = \frac{1}{4} \cdot \frac{8\pi^5}{15} \frac{k^4}{(hc)^3} T^4 \]

\[ = \frac{1}{4} \cdot \frac{2.99792458 \times 10^8}{8\pi^5} \cdot \frac{1380650 \times 10^{23}}{15} \cdot \frac{1}{(6.626079 \times 10^{-34} \cdot \text{c})^3} \]

\[ = \frac{5.6704 \times 10^8}{m^2 \cdot K^4} \]

and so, $R = \sigma T^4$ matches the above expression.

and $\sigma =$ Stefan's constant matches well with

\[ = 5.6703 \times 10^8 \]

\[ = 5.6704 \times 10^8 \]
a) The energy density distribution function: \( u(\lambda) = \frac{8\pi hc \lambda^5}{(e^{\frac{h}{kT \lambda}} - 1)} \) (from Planck).

Let \( \alpha = \frac{hc}{kT} \),

then, \( u = \frac{8\pi hc \lambda^5}{(e^{\frac{h}{kT \lambda}} - 1)} = C\lambda^5 (e^{\frac{h}{kT \lambda}} - 1)^{-1} \) where \( C = 8\pi hc \) as required.

b) To determine \( \lambda \) for which \( \frac{du}{d\lambda} = 0 \):

new, \( u = C\lambda^{-5} (e^{\frac{h}{kT \lambda}} - 1)^{-1} \)

\[ \frac{du}{d\lambda} = -5C\lambda^{-6} (e^{\frac{h}{kT \lambda}} - 1)^{-1} + C\lambda^{-5} \cdot (e^{\frac{h}{kT \lambda}} - 1)^{-2} \cdot \left(-\frac{h}{kT \lambda^2}ight) \cdot e^{\frac{h}{kT \lambda}} \]

when \( \frac{du}{d\lambda} = 0 \),

\[ 5C\lambda^{-6} (e^{\frac{h}{kT \lambda}} - 1)^{-1} = C\lambda^{-5} \left(\frac{h}{kT \lambda^2}\right) e^{\frac{h}{kT \lambda}} \]

\[ 5\lambda (e^{\frac{h}{kT \lambda}} - 1)^{-1} = \frac{\lambda}{e^{\frac{h}{kT \lambda}}} \]

\[ 5\lambda (1 - e^{-\frac{h}{kT \lambda}}) = \alpha \]

\[ 5\lambda (1 - e^{-\frac{h}{kT \lambda}}) = \alpha \], as required.

c) To solve, let \( \lambda = \alpha \alpha \)

\[ \frac{du}{d\lambda} = 0 \Rightarrow 5(\alpha \alpha) \left[1 - e^{-\frac{h}{kT \alpha}}\right] = \alpha \]

\[ 5\alpha \left[1 - e^{-\frac{h}{kT \alpha}}\right] = 1 \]

This can be solved numerically. Using Mathematica, one can use FindRoot command for the function: \( f(x) = 5x(1-e^{-\frac{h}{kT x}}) - 1 \).

\[ \Rightarrow x = 0.201405 \]
$\text{In[1]} := f[a_] := 5a (1 - e^{-1/a}) - 1$

$\text{In[2]} := \text{Plot}[f[x], \{x, 0, 6\}];$

$\text{In[3]} := \text{FindRoot}[f[x] = 0, \{x, 0.3\}]
\text{Out[3]} = \{x \rightarrow 0.201405\}$

$\text{In[4]} := hh = 6.62069 \times 10^{-34}$
\text{Out[4]} = 6.62069 \times 10^{-34}$

$\text{In[5]} := cc = 2.99792458 \times 10^8$
\text{Out[5]} = 2.99792 \times 10^8$

$\text{In[6]} := kk = 1.38065 \times 10^{-23}$
\text{Out[6]} = 1.38065 \times 10^{-23}$

$\text{In[7]} := hh \, cc / kk \ast 0.20140523527264217$
\text{Out[7]} = 0.00289542$

When $a = 0.201405$, then $\lambda = \alpha a$ and $a = \frac{hc}{kt}$

\[\therefore \text{for maximum } u, \frac{du}{d\lambda} = 0 \Rightarrow \lambda = 0.201405 \cdot \frac{hc}{kt}\]

\[\Rightarrow \lambda_T = \frac{0.201405 \cdot hc}{k}\]

\[\Rightarrow \lambda_T = 2.895 \times 10^{-3} \text{ m.K}\]

This compares well with Wien's displacement law

$\lambda_T = 2.898 \times 10^{-3} \text{ m.K}$  or eqs. 3-20 of text.
Consider a light bulb where \( P = 40 \, \text{W} = 40 \, \text{J/s} \)
\( T = 3300 \, \text{K} \)

Assuming light bulb acts like a blackbody,

a) The peak of the blackbody radiates according to Wien's displacement law \( \lambda_m \Gamma = 2.898 \times 10^{-3} \, \text{m.K} \)

\[ \therefore \quad \lambda_m = \frac{2.898 \times 10^{-3}}{3300} = 878.2 \, \text{nm} \]

and so, \( f_m = \frac{c}{\lambda_m} = \frac{2.997924 \times 10^8}{878.2} = 3.414 \times 10^{14} \, \text{Hz} \).

b) The spectral distribution \( R(\lambda) = \frac{2\pi\varepsilon_0 c^2 \lambda^{-5}}{(e^{\lambda / kT} - 1)} \).

Total area:
\[ R = \frac{1}{4} \varepsilon_0 c U = \sigma T^4 \]
\[ = 5.6703 \times 10^{-8} \times (3300)^4 \]
\[ = 6.7245 \times 10^6 \, \text{W/m}^2 \]

\( T = 3300 \, \text{K} \)

\[ \lambda_m \]

\( \lambda \)

\( \int R(\lambda) d\lambda \)

\( \lambda_{\text{visible}} \)

\( \lambda_{700} \)

\( \lambda_{400} \)

\( \lambda \)

\( \sigma \)

\( \varepsilon_0 \)

\( c \)

\( U \)

\( \lambda \)

\( T \)

\( k \)

\( R(\lambda) \)

\( R_{\text{visible}} \)

\( R_{\text{total}} \)

\( R_{\text{700nm}} \)

\( R_{\text{400nm}} \)

\( 0 \)

\( \infty \)

\( 0 \)

\( \lambda_{700} \)

\( \lambda_{400} \)

\( \int_{\lambda_{400}}^{\lambda_{700}} R(\lambda) d\lambda \)

\( \int_0^{\lambda_{700}} R(\lambda) d\lambda \)

\( \int_0^{\lambda_{400}} \frac{2\pi\varepsilon_0 c^2 \lambda^{-5}}{(e^{\lambda / kT} - 1)} d\lambda \)

\( \int_0^{\lambda_{700}} \frac{2\pi\varepsilon_0 c^2 \lambda^{-5}}{(e^{\lambda / kT} - 1)} d\lambda \)

\[ = \frac{700 \times 2\pi\varepsilon_0 c^2 \lambda^{-5}}{6 \times 7245 \times 10^6} = 0.1175 \]

\[ \therefore \quad \approx 12\% \text{ of radiated energy is in the visible range.} \]

(not very efficient, but as you can see from the information on the attached webpage, attempting to increase this temperature compromises the strength of the tungsten filament.)
d) Assuming $f_m$ is a good approximation for the frequency correspondence to the average photon energy, then,  
$$E_{ph} = hf = 6.626079 \times 10^{-34} \cdot 3.414 \times 10^{14} = 2.262 \times 10^{-19} J$$  
($= 1.41 \text{eV}$)

To determine n° of photons emitted, 
know Power = 40 J/sec

\[ \therefore \text{in every sec, n° of emitted photons} = \frac{40}{2.262 \times 10^{-19}} = 1.77 \times 10^{20} \]  
(i.e. lots!) 

e) 
\[ \text{Area pupil} = \pi \left(2.5 \times 10^{-3}\right)^2 \]
\[= 6.25 \times 10^{-8} \]

\[\therefore \text{n° of photons entering eye} = 6.25 \times 10^{-8} \times 1.77 \times 10^{20} \]
\[= 1.1 \times 10^{13} \text{eye/second} \]  
(still lots)
Luminous Efficiency

In a 120 volt, 100 watt "standard" bulb with a rated light output of 1750 lumens, the efficiency is 17.5 lumens per watt. This compares poorly to an "ideal" of 242.5 lumens per watt for one idealized type of white light, or 683 lumens per watt ideally for the yellowish-green wavelength of light that the human eye is most sensitive to.

Other types of incandescent light bulbs have different efficiencies, but all generally have efficiencies near or below 35 lumens per watt. Most household incandescent bulbs have efficiencies from 8 to 21 lumens per watt. Higher efficiencies near 35 lumens per watt are only achieved with photographic and projection lamps with very high filament temperatures and short lifetimes of a few hours to around 40 hours.

The reason for this poor efficiency is the fact that tungsten filaments radiate mostly infrared radiation at any temperature that they can withstand. An ideal thermal radiator produces visible light most efficiently at temperatures around 6300 Celsius (6600 Kelvin or 11,500 degrees Fahrenheit). Even at this high temperature, a lot of the radiation is either infrared or ultraviolet, and the theoretical luminous efficiency is 95 lumens per watt.

Of course, nothing known to any humans is solid and usable as a light bulb filament at temperatures anywhere close to this. The surface of the sun is not quite that hot.

There are other ways to efficiently radiate thermal radiation using higher temperatures and/or substances that radiate better at visible wavelengths than invisible ones. This is covered by Part II of the Great Internet Light Bulb Book, Discharge Lamps. The efficiency of an incandescent bulb can be increased by increasing the filament temperature, which makes it burn out more quickly.

http://members.misty.com/don/bulb1.html
At top of earth's atmosphere, sun delivers $1.36 \times 10^3$ W/m$^2$. [Solar constant]

Assuming the earth radiates like a blackbody at uniform temperature, known,

$$ R = \sigma T^4 $$

power radiated per unit area.

However, we are only given the power absorbed by the earth per unit area. These two quantities are different because earth absorbs energy only on side facing the sun (ie total area = $\pi R_e^2$). However, a perfect blackbody will radiate in all directions ($\Rightarrow$ energy is emitted over all surface area, $4\pi R_e^2$ where $R_e$ = earth's radius.

$$ \text{Power radiated by earth per unit area} = \frac{1.36 \times 10^3 \times \pi R_e^2}{4\pi R_e^2} = 3.4 \times 10^2 \text{ W/m}^2 $$

now,

$$ R_e = 3.4 \times 10^7 = \sigma T_e^4 $$

$$ T_e^4 = \frac{3.4 \times 10^7}{5.67 \times 10^{-8}} \Rightarrow T_e = 278.3 \text{ K} $$

$$ = 51 \text{ °C} \quad (T_{obs} = 273.15 \text{ K}) $$
Q7: Consider a given intensity and frequency of light incident on a sample of sodium.

\[ I_2 \approx 2I, \]

(a) The current levels off at high values of anode voltage. This corresponds to the case where all emitted photoelectrons are collected by the collector plate. There are no more photoelectrons unaccounted for in the process, hence, the current levels off.

(b) When the intensity of the incident light is doubled to twice the original intensity,

i) The total mass of photocurrent increases

ii) The stopping potential remains unchanged.

j) Increasing intensity increases the overall number of photons striking the sodium, however, each photon has the same incident energy as before. Therefore, number of photoelectric events increases, but \( V_{\text{stop}} \) corresponding to the maximum kinetic energy of the released electron, does not change.

(c) When the frequency is doubled, the energy of each photon increases (also doubled) \( \Rightarrow \) maximum kinetic energy of released electron is greater \( \Rightarrow V_{\text{stop}} \) occurs at a more negative voltage. However, the total number of photons will decrease because total energy/sec is the same and each photon having more energy means there will be fewer emitted per second.
Under optimal conditions, to perceive a flash need ~60 photons at the cornea.

If $\lambda \sim 550 \text{ nm (very average)}$

$$E_{ph} = \frac{hc}{\lambda} = \frac{1239.8 \text{ eV.nm}}{550 \text{ nm}} = 2.254 \text{ eV}$$

:. 60 photons of this wavelength corresponds to $60 \times 2.254 \text{ eV} = 135 \text{ eV}$

$\sim 2.17 \times 10^{-17} \text{ Joule!}$

(amazing - but true!)
Given data of stopping potential, \( V_0 \), as a function of incident wavelength for the photoelectric effect, we know:

\[
E_{ph} = \phi + eV_0
\]

\[
\frac{hc}{\lambda} = \phi + eV_0
\]

\[
\therefore eV_0 = \frac{hc}{\lambda} - \phi
\]

\[
\therefore V_0 = \frac{hc}{e\lambda} - \frac{\phi}{e}
\]

- Plotting \( V_0 \) against \( \frac{1}{\lambda} \):

a) By plotting \( \frac{1}{V_0} \) and obtaining the y-intercept from a straight-line fit will give the work function (in eV).

\[
\phi = 2.085 \text{ eV}
\]

b) The threshold frequency can be obtained by finding where the above graph crosses the \( V_0 = 0 \) (x-axis).

Now, from a fit using Mathematica,

\[
V_0 = -2.08545 + 1.25361 \times 10^{-6} \left( \frac{1}{\lambda} \right)
\]

when \( V_0 = 0 \) (electron released with no kinetic energy)

\[
2.08545 = 1.25361 \times 10^{-6} / \lambda \quad \Rightarrow \quad \lambda = 6.01118 \times 10^{-7} \text{ m}
\]

\[
\Rightarrow f_{\text{max}} = \frac{c}{\lambda} = 4.987 \times 10^{14} \text{ Hz}
\]

c) Slope of graph = \( \frac{hc}{e} \)

\[
\therefore \frac{hc}{e} = \text{slope} / c = \frac{1.25361 \times 10^{-6}}{2.99792458 \times 10^8} = 4.182 \times 10^{-15} \text{ eV s}
\]
In[18]:= llist = {200, 300, 400, 500, 600} * 10^(-9);
vlist = {4.20, 2.06, 1.05, 0.41, 0.03};

In[38]:= (*created a list of coordinate pairs
   {1/wavelength, V_o} *)

In[26]:= lvlist = Transpose[{1/llist, vlist}]

Out[26]= {{5000000, 4.2}, {10000000/3, 2.06}, {2500000, 1.05}, {2000000, 0.41}, {5000000/3, 0.03}}

In[29]:= ListPlot[lvlist, PlotStyle -> PointSize[0.02]];

In[36]:= (* to perform a linear fit to the data *)

In[24]:= Fit[lvlist, {1, x}, x]

Out[24]= -2.08545 + 1.25361*10^-6 x

In[35]:= (*To obtain wavelength at V_stop = 0 *)

In[30]:= 1.2536050156739808^-6 / 2.085454545454545


In[34]:= (*to obtain threshold frequency*)

In[31]:= cc / 6.01183597390493^-7

Out[31]= 4.98725*10^14

In[33]:= (* to obtain ratio h/e *)

In[32]:= 1.2536050156739808^-6 / cc

Out[32]= 4.18158*10^-15