In the case of Compton scattering,

\[ E_i \quad \rightarrow \quad 0 \]

Assuming X-ray acts like a particle of energy \( E = hf \) and momentum \( p = \frac{E}{c} \), for an elastic collision, both energy and momentum are conserved.

Conservation of momentum:

\[ P_e \rightarrow P_2 \]

\[ \vec{P}_i = \vec{P}_e + \vec{P}_2 \]

First, \( \vec{P}_e = (\vec{P}_1 - \vec{P}_2) \)

\[ P_e^2 = |\vec{P}_e|^2 = |(\vec{P}_1 - \vec{P}_2)|^2 = |\vec{P}_1^2 + \vec{P}_2^2 - 2\vec{P}_1 \cdot \vec{P}_2| \]

\[ P_e^2 = P_1^2 + P_2^2 - 2p_1p_2 \cos \theta \tag{1} \]

Conservation of energy:

\[ (E_i = P_e^2 + (mc^2)^2) \]

\[ E_i + E_0 = E_2 + \sqrt{E_0^2 + (P_e)^2} \quad \text{where} \quad E_0 = mc^2 \]

Rearranging (2):

\[ E_0 + (E_i - E_2) = \sqrt{E_0^2 + (P_e)^2} \]

Squaring both sides:

\[ E_0^2 + (E_i - E_2)^2 + 2E_0(E_i - E_2) = E_0^2 + (P_e)^2 \]

\[ (P_e)^2 = [(p_1c) - (p_2c)]^2 + 2E_0(p_1c - p_2c) \]

\[ p_e^2 = p_1^2 + p_2^2 - 2p_1p_2 + \frac{2E_0}{c} (p_1 - p_2) \tag{2} \]

Equating expressions for \( P_e^2 \) in (1) and (2)
\[
\begin{align*}
\frac{p_1^2 + p_2^2 - 2p_1 p_2 \cos \theta}{E_0} &= \frac{p_1^2 + p_2^2 - 2p_1 p_2 + 2E_0}{E_0} (p_1 - p_2) \\
&= \frac{p_1 p_2}{E_0} (1 - \cos \theta) = \frac{E_0}{c} (p_1 - p_2) \\
\therefore \quad 1 - \cos \theta &= \frac{E_0}{c} \left( \frac{p_1 - p_2}{p_1 p_2} \right) = \frac{E_0}{c} \left( \frac{1}{p_2} - \frac{1}{p_1} \right) ; \quad \text{let} \quad E_1 = \frac{p_1 c}{\lambda_1} - \frac{hc}{\lambda_2} \quad \text{and} \quad E_2 = \frac{p_2 c}{\lambda_2} = \frac{hc}{\lambda_2} \\
\equiv \quad 1 - \cos \theta &= \frac{E_0}{c} \left( \lambda_2 - \lambda_1 \right) \\
&= \frac{hc}{mc} \left( 1 - \cos \theta \right) = \frac{h}{mc} (1 - \cos \theta) \\
\equiv \quad \lambda_2 &= \lambda_1 + \frac{h}{mc} (1 - \cos \theta) \\
To \ determine \ the \ kinetic \ energy \ of \ the \ electron,
\quad E_k &= E_1 - E_2 = E_1 - \frac{hc}{\lambda_2} \\
&= \frac{E_1}{\lambda_1} - \frac{hc}{\lambda_1} \left( \frac{1}{\lambda_1 + \frac{h}{mc} (1 - \cos \theta)} \right) \\
&= \frac{hc}{\lambda_1} - \frac{hc}{\lambda_1} \left( \frac{1}{\lambda_1 + \frac{h}{mc} (1 - \cos \theta)} \right) \\
&= \frac{hc}{\lambda_1} \left[ \frac{1}{\lambda_1 + \frac{h}{mc} (1 - \cos \theta)} - \frac{\lambda_1}{\lambda_1} \right] \\
&= \frac{hc}{\lambda_1} \left[ \frac{1}{\lambda_1 + \frac{h}{mc} (1 - \cos \theta)} - \frac{\lambda_1}{\lambda_1 + \frac{h}{mc} (1 - \cos \theta)} \right] \\
&= \frac{hc}{\lambda_1} \left( \frac{1}{\lambda_1 + \frac{h}{mc} (1 - \cos \theta)} \right) \\
E_1 &= \frac{hc}{\lambda_1} \left( \frac{1}{\lambda_1 + \frac{h}{mc} (1 - \cos \theta)} \right) \\
&= \frac{E_1}{\lambda_1} \frac{h}{mc} (1 - \cos \theta) \\
&= \frac{E_1}{\lambda_1} \frac{h}{mc} (1 - \cos \theta)
\end{align*}
\]
\[ E_k = E_1 \cdot \frac{\frac{h}{m_e c} (1 - \cos\Theta)}{\frac{hc}{E_1} + \frac{h}{m_e c} (1 - \cos\Theta)} \]

\[ = E_1 \cdot \frac{\frac{k}{m_e c} (1 - \cos\Theta)}{K\left[ \frac{1}{E_1} + \frac{1}{E_0} (1 - \cos\Theta) \right]} \quad \text{where} \quad \frac{E_0}{E_1} = \frac{m_e c^2}{\text{rest mass energy of electron}} \]

\[ = \frac{E_1}{E_0} \cdot (1 - \cos\Theta) \cdot \frac{E_1 E_0}{E_0 + E_1 (1 - \cos\Theta)} \]

\[ = \frac{E_1^2 (1 - \cos\Theta)}{E_0 + E_1 (1 - \cos\Theta)} = E_1 \times \left\{ \frac{E_1 (1 - \cos\Theta)}{E_0 + E_1 (1 - \cos\Theta)} \right\} \]

\[ = \frac{E_1 (1 - \cos\Theta)}{E_0/E_1 + (1 - \cos\Theta)} \]

\[
\text{new, } E_k = \frac{E_1}{\frac{E_0}{E_1 (1 - \cos\Theta)} + 1}
\]

when \( \Theta = 180^\circ \)

\[ \Rightarrow E_k = \frac{E_1}{\frac{E_0}{2E_1} + 1} \]

\[ = \frac{E_1}{1 + \frac{E_0}{2E_1}} \]

\[ = \frac{hf}{1 + \frac{mc^2}{2hf}} \quad \text{where } E_1 = hf \quad \text{and } \frac{E_0}{m_e c^2} \]

\[ \Rightarrow \text{denominator is smallest} \]

\[ \Rightarrow (1 - \cos\Theta) \text{ term is largest} \]

\[ \Rightarrow \cos\Theta = -1 \quad \text{and } \Theta = 180^\circ \]

\[ \Rightarrow \text{complete backscattering of photon.} \]

\[ \text{to determine the maximum recoiling kinetic energy of the electron, want } E_k \text{ to be a maximum} \]

\[ \text{as required.} \]
(* For Compton scattering: *)

(* lambda2 = lambda1 + h/(m c) [1 - cos(theta)] *)

h = 6.625 * 10^(-34); (*J s*)

m = 9.109 * 10^(-31); (*kg*)
c = 2.998 * 10^8; (*m/s*)
hc = 1.240 * 10^3 (*eV nm*);
e = 1.602 * 10^(-19) (*Coulomb*);

E0 = m c^2/e (*rest mass energy of electron (eV) *)


Einc = 520 * 10^3 (*eV*);

lbdal = hc/Einc (*in nm*)

Out[8] = 0.00238462

h / (m c)

Out[9] = 2.42596 * 10^{-12}

lbdal = El, th := hc/El + h / (m c) * 10^9 * (1 - Cos[th*180/Pi]) (*in nm*)

Out[10] = 0.00702351

(*To obtain an expression for the kinetic energy of the recoiling electron; electron recoil energy = difference between incident and scattered photons*)

Ekin[El_, th_] := El (1 - Cos[th*180/Pi]) / (E0 / El + (1 - Cos[th*180/Pi])) (*in eV*)

Out[11] = 0.00702351

(*To generate Compton scattering of Einc but at random angles between 0-360 degrees*)


(* Creating a For loop that randomly Compton scatters *)

Out[13] = (empty list to contain kinetic energy of recoiled electron*)

Out[14] = For[j = 1, j <= 1000, j = j + 1,

ww = Ekin[Einc, Random[Real, {0, 360.}]];

AppendTo[outlist, ww]

] (*end For*)

Out[15] = outlist; (*used for trouble-shooting*)

Out[16] = (*load in Graphics package to enable plotting of histograms*)

In[22]:= Histogram[ourlist];
Q3  Galactic Red Shift

Spectral wavelengths of 4117 Å and 4357 Å for hydrogen corresponds closely to the \( \text{H}_\gamma \) and \( \text{H}_\delta \) lines of the Balmer series. From the Rydberg-Ritz formula,

\[
\frac{1}{\lambda_{mn}} = R_\infty \left( \frac{1}{n^2} - \frac{1}{m^2} \right) \text{ for } n > m \quad \text{where} \quad R_\infty = 109,677.6 \times 10^7 \text{ m}^{-1}
\]

\( \text{H}_\gamma \) corresponds to \( n=5 \rightarrow m=2 \) \( \therefore \lambda_\gamma = 4341.7 \text{ Å} \)

\( \text{H}_\delta \) \( n=6 \rightarrow m=2 \) \( \therefore \lambda_\delta = 4102.7 \text{ Å} \)

The apparent shift in wavelength are then

\[
\Delta \lambda_\gamma = 4341.7 - 4357 = 15.3 \text{ Å}
\]

\[
\Delta \lambda_\delta = 4102.9 - 4117 = 14.1 \text{ Å}
\]

For light, the Doppler effect is given by:

\[
f_{\text{obs}} = \sqrt{\frac{1 + \beta}{1 - \beta}} f_{\text{source}} \quad \text{for } (\text{approaching})
\]

\[
f_{\text{obs}} = \sqrt{\frac{1 - \beta}{1 + \beta}} f_{\text{source}} \quad \text{for } (\text{receding})
\]

For our case, the shift is to make the characteristic wavelengths longer,

\( \Rightarrow \) receding source

\[
\frac{hc}{\lambda_{\text{obs}}} = \sqrt{\frac{1-\beta}{1+\beta}} \cdot \frac{hc}{\lambda_{\text{source}}}
\]

\[
\therefore \lambda_{\text{obs}} = \sqrt{\frac{1+\beta}{1-\beta}} \lambda_{\text{source}} \quad \text{and so,} \quad \sqrt{\frac{1+\beta}{1-\beta}} = \left\{ \begin{array}{l}
1.0035 \lambda_\gamma \\
1.0034 \lambda_\delta
\end{array} \right.
\]

\( \Rightarrow 1 + \beta = 1.00706 (1 - \beta) \)

\( \Rightarrow 2.00706 \beta = (1.00706 - 1) \)

\( \Rightarrow \beta = 3.52 \times 10^{-3} = \frac{v}{c} \) \( \Rightarrow v = 1.05 \times 10^6 \text{ m/s} \approx 10,000 \text{ km/s} \)
In each collision, a particle is deflected by $0.01^\circ$.

To obtain an average (rms - root mean squared) deflection of $10^\circ$, recall

$\langle \theta \rangle = \sqrt{N} \theta_{\text{indiv}}$

$\therefore 10^\circ = \sqrt{N} \times (0.01^\circ)$

$\Rightarrow \sqrt{N} = 10^3$

$\therefore N = 10^6$ i.e. need $\sim$1 million collisions to obtain an average deflection of $10^\circ$.

To determine the number of atomic layers in a gold foil of $10^{-6}$ m thickness:

Let $t = 10^4$ m (thickness

$t_{\text{atom}} = 10^{-10}$ m (thickness of each atom)

$\therefore \text{no of atomic layers} = \frac{10^{-6}}{10^{-10}} = 10^4$

Here, the gold thickness is approximately 10,000 atomic layers thick. This is 100 times less than the thickness required to obtain an average deflection of $10^\circ$.

In fact,

$\langle \theta \rangle = \sqrt{10^4} \theta_{\text{indiv}} = 1^\circ$ i.e. average deflection is about $1^\circ$.
Consider a sample of hydrogen atoms are all in the $n=5$ state. If all atoms eventually return to the ground state and that all transitions are possible, need to consider the energy level diagram for hydrogen.

Consider a sample of 500 atoms. There is equal probability for each possible transition.

Doing this in general, let $N$ total no. of atoms in $n=5$ state in sample.

Tracking the paths to the ground state: There are 10 distinct photon energies.

- $n=5 \rightarrow n=1$ (directly) \[\text{no. of photons} = \frac{N}{4}\]

- $n=5 \rightarrow n=2$
  
  $n=2 \rightarrow n=1$
  \[\text{no. of photons} = \frac{N}{4} \times 2\]

  two photons emitted in this process.

  50% prob of generating 1 photon
  50% prob of generating 2 photons

  \[= \frac{N}{4} \times 2\]

- $n=5 \rightarrow n=4$
  
  \[\text{no. of photons} = \frac{N}{4} \sum_{\substack{s=2,3 \rightarrow 1 \text{ prob } \frac{1}{3} \ \ \frac{2}{3} \ \ \frac{1}{3} \ \ \frac{4}{3} \ \ \frac{4}{3} \ \ \frac{4}{3} \ \ \frac{1}{3}}} + \frac{1}{3} \times 1 + \frac{1}{3} \times 2 + \frac{1}{3} \times \left(1+\frac{1}{3}+\frac{2}{3}\right)\]

  Total no. of photons = $\frac{N}{4} \left[1+2+\frac{5}{3}+\frac{17}{24}\right] = \frac{25}{12} N = 104.2$ photons