Natural and Step Responses for RLC Circuits

The natural and step responses of RLC circuits are described by second-order, linear differential equations with constant coefficients and constant "input" (or forcing function),

$$a \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + cx(t) = D,$$  \hspace{1cm} (1)

where $a, b, c,$ and $D$ are constants, and the initial values $x(0^+)$ and $\frac{dx(0^+)}{dt}$ are known (these are found by circuit analysis). We assume $a > 0$, without loss of generality.

As you know from MATH 212, the general solution for $t > 0$ is

$$x(t) = x_c(t) + x_p(t),$$ \hspace{1cm} (2)

where $x_c(t)$ is the complementary solution to the homogeneous equation and $x_p(t)$ is a particular solution. The particular solution is

$$x_p(t) = \begin{cases} \frac{D}{c}, & c \neq 0 \\ \frac{D}{b} t, & c = 0, b \neq 0 \\ \frac{D}{2a} t^2, & c = b = 0 \end{cases}.$$ \hspace{1cm} (3)

The homogeneous differential equation is

$$a \frac{d^2 x_c}{dt^2} + b \frac{dx_c}{dt} + cx_c(t) = 0.$$ \hspace{1cm} (4)

Its solution is determined by the characteristic (or auxiliary) equation,

$$as^2 + bs + c = 0,$$ \hspace{1cm} (5)

which has roots

$$s_1 = -\frac{b}{2a} + \sqrt{\left( \frac{b}{2a} \right)^2 - \frac{c}{a}} = -\alpha + \sqrt{\alpha^2 - \omega_0^2},$$ \hspace{1cm} (6)

$$s_2 = -\frac{b}{2a} - \sqrt{\left( \frac{b}{2a} \right)^2 - \frac{c}{a}} = -\alpha - \sqrt{\alpha^2 - \omega_0^2}.$$ \hspace{1cm} (7)

The neper frequency is $\alpha = b/(2a)$ and the resonant frequency is $\omega_0 = \sqrt{c/a}$ (in radians). If $c < 0$, then $\omega_0$ is imaginary and $s_1 > 0$ in (6).
There are three cases.

1. Real and unequal roots (overdamped): \( \alpha^2 > \omega_0^2 \)

\[
x_c(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}
\]  

Both terms are *decaying* exponentials if \( c > 0 \).

2. Real and equal roots (critically damped): \( \alpha^2 = \omega_0^2 \)

\[
x_c(t) = c_1 e^{s_1 t} + c_2 t e^{s_1 t}, \text{ where } s_1 = -\alpha = -\frac{b}{2a}
\]  

3. Complex roots (underdamped): \( \alpha^2 < \omega_0^2 \)

The roots have the form

\[
s_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j \omega_d
\]  

where \( \omega_d \) is the damped frequency (in radians). Then

\[
x_c(t) = c_1 e^{-\alpha t} \cos(\omega_d t) + c_2 e^{-\alpha t} \sin(\omega_d t)
\]

In the case that \( \alpha = 0 \), then \( \omega_d = \omega_0 \) and the solution is pure, undamped oscillation,

\[
x_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).
\]

This is why \( \omega_0 \) is called the resonant frequency.

Summary:

- The general solution is in equation (2): \( x(t) = x_c(t) + x_p(t) \).
- The complementary solution \( x_c(t) \) is either (8), (9), or (11).
- The particular solution \( x_p(t) \) is in (3). For the natural response, \( D = 0 \) and \( x_p(t) = 0 \).
- The constants \( c_1 \) and \( c_2 \) are determined from the initial conditions \( x(0^+) \) and \( \frac{dx(0^+)}{dt} \).
- Hints for finding initial conditions in RLC circuits:
  - Parallel RLC: Analyze voltages, so \( x(t) = v(t) \). Then \( v(0^+) \) is determined from the initial capacitor voltage, and \( \frac{dv(0^+)}{dt} \) is determined from KCL and the initial inductor current.
  - Series RLC: Analyze currents, so \( x(t) = i(t) \). Then \( i(0^+) \) is determined from the initial inductor current, and \( \frac{di(0^+)}{dt} \) is determined from KVL and the initial capacitor voltage.