Chapter 2

Graphs

A Friendly Introduction for Electrical and Computer Engineers

Probability and Stochastic Processes
Random Variables

Experiment: Procedure + Observations

Observation is an outcome

Assign a number to each outcome: Random variable

Random Variables
Three ways to get a RV:

- The RV is a function of a RV
- The RV is a function of the observation
- The RV is the observation

Random Variables
Discrete Random Variables

- Discrete RV $X$ has PMF $p_X$
- $X$ is discrete if $\mathcal{X}$ is countable
- Range of $X$ ($\mathcal{X}$, set of possible values) $\subseteq \mathbb{X}$

$[x = X]d = (x)^x d$
\( (x)^P \bigcup_{B \in X} \left[y \in X \right] p = [B \in X] p = [B] p \)

For an event \( B \subseteq X \) we have

\[ I = (x)^P \bigcup_{x \in X} x \left[ y \in X \right] p \]

\[ 0 < (x)^P \]

PMF Properties
Bernoulli RV

Get the phone number of a random student. Let $X = 0$ if the last digit is even. Otherwise, let $X = 1$.

$$
\begin{align*}
1 &= x & d \\
0 &= x & d - 1
\end{align*}
= (x)^1 (1 - d)
$$
Binomial RV

circuits, each circuit is rejected with probability independent of other tests.

\[
\begin{align*}
\text{no. of rejects} &= Y \\
\text{otherwise} \quad u', \ldots, 1', 0 = \kappa \quad \gamma - u(d - 1) \gamma d(\gamma, u) \\
\end{align*}
\]

\[= (\kappa) \quad \gamma d\]

\[Y \] is the number of successes in \( n \) trials.

\[\text{no. of rejects} = Y \]

\[\text{other tests.}\]

\[\text{Test } n \text{ circuits, each circuit is rejected with probability independently of other tests.}\]
\[ \begin{cases} 1 = \hat{f} & 0 \\ \hat{f}(d - 1) \end{cases} \}

\begin{align*}
(d - 1) \hat{d} &= [\hat{z} = \lambda] \hat{d}, \\
\hat{d} &= [\hat{1} = \lambda] \hat{d}.
\end{align*}

From the tree, \( d = \alpha \)

\begin{itemize}
    \item \( v \rightarrow d \rightarrow \hat{d} \)
    \item \( \hat{1} = \lambda \)
    \item \( \hat{z} = \lambda \)
\end{itemize}

First reject.

Circuit rejected with prob \( \lambda \). \( \hat{d} \) is the number of tests up to and including the

**Geometric RV**
otherwise \( 0 \leq \alpha \leq 2 \),

\[
\left\{ \begin{array}{l}
0 \\
(1 - \alpha)(8 \cdot 0.2)
\end{array} \right.
\]

\[
\Rightarrow (\alpha) \chi_d
\]

Geometric: 0.2 = \( d \)
Events $A$ and $B$ are independent

$\{ \text{success on attempt } i \} = B$

$\{ \text{success in } i - 1 \text{ tests} \} = A$

$[B \land A] d = [i = T] d$

No. of tests, $T$, needed to find $k$ rejects.

Pascal RV
otherwise
\[
\begin{cases}
\ldots \cdot \cdot \cdot + \eta, \eta = 1 & \eta \cdot (d - 1) \cdot 2d \left( \frac{1 - \eta}{1 - \eta} \right) \\
0 & \\
\end{cases}
\]
\[
[B]d [A]d = (I) \cdot d
\]
\[
(1 - \eta) \cdot (d - 1) \cdot 2d \left( \frac{1 - \eta}{1 - \eta} \right) = \\
[A]d = [A]d
\]

\[\text{Pascal continued}\]
otherwise

\[
\begin{array}{c}
\forall \ell = 1 \quad \ell \cdot (0.8 \cdot 0, 0, 0, 0) \quad (\ell - 1, 1)
\end{array}
\]
• Pascal
• Geometric
• Binomial
• Bernoulli

Summary
\begin{align*}
\text{otherwise} & \quad 0 = x \\ & \quad |x/\lambda - x| \\
\end{align*}
\begin{align*}
\begin{cases}
0 & \\
(\lambda x)^x I
\end{cases}
\end{align*}
\text{With } \alpha = xL.
\begin{itemize}
\item Arrival rate \( \lambda \), interval time \( T \).
\item Counts arrivals of something.
\end{itemize}
\textbf{Poisson RV}
Poisson: \( \alpha = 0.5 \)

\[
P_j(j) = \begin{cases} 
(0.5)^j e^{-0.5}/j! & \text{if } j = 0, 1, \ldots \\
0 & \text{otherwise}
\end{cases}
\]
\[ P(j) = \frac{e^{-\lambda} \lambda^j}{j!} \]

\[ j = 0, 1, 2, \ldots \]

\[ \sum_{j=0}^{\infty} P(j) = 1 \]

\[ \sum_{j=0}^{\infty} \frac{e^{-\lambda} \lambda^j}{j!} = 1 \]

\[ \int_{\lambda}^{\infty} \frac{e^{-\lambda} \lambda^j}{j!} \, d\lambda = 0 \]

\[ \int_{0}^{\infty} \frac{e^{-\lambda} \lambda^j}{j!} \, d\lambda = \frac{1}{j!} \]

\[ \sum_{j=0}^{\infty} \frac{e^{-\lambda} \lambda^j}{j!} = 1 \]

\[ \int_{0}^{\infty} e^{-\lambda} \lambda^j \, d\lambda = \frac{\lambda^j}{j!} \]

\[ \sum_{j=0}^{\infty} \frac{e^{-\lambda} \lambda^j}{j!} = 1 \]

\[ \sum_{j=0}^{\infty} P(j) = 1 \]

\[ \sum_{j=0}^{\infty} \frac{e^{-\lambda} \lambda^j}{j!} = 1 \]

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\[ \int_{0}^{\infty} e^{-\lambda} \lambda^j \, d\lambda = \frac{1}{j!} \]
The cumulative distribution function (CDF) of random variable $X$ is

$$[x \geq X]d = (x)^X P$$

Cumulative Distribution Functions
At the discontinuities $r$ is the upper value. (right hand limit)
CDF Properties

For any discrete RV $X$, range $X$, satisfying $\{ \ldots, x, x, \ldots \} = X$.

\begin{align*}
1 + x &> x \geq \frac{x}{2} \text{ for } (\frac{x}{2})X_H = (x)X_H \quad \bullet \\
(\frac{x}{2})X_H &= (\varepsilon - \frac{x}{2})X_H - (\frac{x}{2})X_H \\
\text{For all } x \in \mathbb{R} \text{ and small } \varepsilon > 0 \quad \bullet \\
(x)X_H &< (\frac{x}{2})X_H, x < \frac{x}{2} \quad \bullet \\
1 &= (\infty)X_H \text{ and } 0 = (\infty-)X_H \quad \bullet \\
\ldots \geq \frac{x}{2} \geq 1 \quad \text{satisfying } \{ \ldots, x, x, \ldots \} = X \text{ and } X \text{ range } X.
\end{align*}
The expected value of $X$ is also called the average of $X$. 

\[
\begin{align*}
\sum_{x \in X} x p(x) &= x \cdot 1 = [X] \mathbb{E} \\
\text{The expected value of } X \text{ is } &\text{Expected Value}
\end{align*}
\]
Average of samples:

If each occur $m$ times,

$\sum_{x \in X} x = \sum_{x \in X} \sum_{u \in U} x = \sum_{u \in U} \sum_{x \in X} x = \sum_{u \in U} \sum_{x \in X} x$

Each $x \in X$ occurs $m$ times.

Average of $u$ samples:

$[X] \in \mathbb{E}$

Average vs.
Each sample value of a derived RV is a function \( g(x) \) of a sample value \( x \) of a RV \( X \).

**Notation:**

\( (X)g = Y \)

**Experimental Procedure**

1. Perform experiment, observe outcome.

2. Find \( x \), the value of \( X \).

3. Calculate \( (x)g \).
\[ \hat{\alpha} = (x) \delta : x \]
\[
\left( x \right)^\chi_D \subset \subseteq \ = \left( \hat{\alpha} \right)^\chi_D
\]
\[
\left( \hat{X} \right) \delta = \lambda \ j^0 \ \text{PMF}
\]
Problem 2.6.5

Source transmits data packets to receiver. If received packet is error-free, receiver sends back ACK, otherwise NAK sent. For each NAK, the packet is resent. Each packet transmission is independently corrupted with prob \( p \).

Find the PMF of \( X \), no. of times a packet is sent. Each packet takes 1 msec to transmit. Source waits 1 msec to receive ACK. \( T \) equals the time req'd until the packet is received OK. What is \( \mathbb{P}(T) \)?

Source transmits data packets to receiver.
$q + [X] \mathbb{E} p = (x) \sum_{x \in X} x f(x) \mathbb{E} = [\lambda] \mathbb{E}$

Example: $q + Xp = \lambda$.

$\sum_{x \in X} (x) f(x) \mathbb{E} = \lambda \mathbb{E} = [\lambda] \mathbb{E}$

Thm: Given RV $X$ with PMF $p$, the expected value of $(X)^\delta$ is $\lambda$.

$\mathbb{E}$
Variance and Standard Deviation

Units of \( \mathcal{X} \) are the same as \( X \).

\[
[X]_{\text{Var}} = \mathcal{X} \quad \text{Standard Deviation}
\]

Variance measures spread of PMF

\[
[X]_{\text{Var}} = \left( \sum_{x \in \mathcal{X}} x \mathcal{P}(x \mathcal{M} - X) \right) = [\lambda] \mathcal{E}
\]

\[
\mathcal{P}(x \mathcal{M} - X) = \lambda \quad \text{Variance}
\]

Variance and Std Deviation
Properties of the variance

\[ [X]^{\text{Var}} \cdot \varphi = [\lambda]^{\text{Var}} \cdot X' \cdot \lambda = \lambda \cdot I \]

\[ [X]^{\text{Var}} = [\lambda]^{\text{Var}} \cdot q + X = \lambda \cdot I \]
Two kinds of conditioning.

\[ [B \mid x = X] d = (x) \frac{B \mid X}{P} \]

Given \( B \), with \( P_B > 0 \).

Conditional PMF of \( X \) given \( B \)
Probability model tells us $P_X | B^z (x)$ for possible $B^z$. Example: In the $i$th month of the year, the number of cars crossing the GW bridge is Poisson with parameter $\lambda$. Conditional PMFs - Version 1
Conditional PMFs - Version 2

is an event defined in terms of

\[
\begin{align*}
B \ni x \quad &\left\{ \begin{array}{l}
0 & \text{otherwise} \\
\frac{[B]d}{(x)\times d} & \text{if } B \ni x \ni X \ni X
\end{array} \right. \\
\end{align*}
\]

\[
= \frac{[B]d}{[B, x = X]d} = (x)_{(X, X)}d
\]

\( B \) is a subset of \( S \) such that for each \( x \in S \), either \( x \in B \) or \( x \notin B \).

\( B \) is an event defined in terms of \( X \).
Example: $X$ is geometric with $p = 0.1$. What is the conditional PMF of $X$ given event $B$ that $X < 9$?
\[ \mathbb{E} \left( r([B|X]\mathbb{E} - X) \right) \mathbb{E} = [B|X] \mathbb{E} \]

\( (x) \frac{d}{dx} \mathbb{E} \mathbb{E} \mathbb{E} = [B|(X)\mathbb{E}] \mathbb{E} \)

\( (x) \frac{d}{dx} \mathbb{E} \mathbb{E} \mathbb{E} = [B|X] \mathbb{E} \)

(\( q|x \)) \frac{d}{dx} \mathbb{E} \mathbb{E} \mathbb{E} = \text{Replace } (x) \) \( X \) \( P \) with (\( x \)) \( X \) \( P \) \( \text{Conditional Expectations} \)