Homework 10

Date Assigned: Thursday, April 10, 2003
Date Due: Tuesday, April 22, 2003

Reading: Please read Chapter 4, Sections 4.1 - 4.5, for April 17 and the week of April 21.

1. In this problem, I would like you to perform an analysis of a digital communication system that is similar to the case that we studied in class. The only difference is that the signal \( S \) and noise \( N \) have PMFs as follows.

\[
P_S(s) = \begin{cases} 
0.25, & s = -1 \\
0.75, & s = +1 \\
0, & \text{otherwise}
\end{cases}
\]

\[
P_N(n) = \begin{cases} 
0.5, & n = -1 \\
0.25, & n = 0 \\
0.25, & n = +1 \\
0, & \text{otherwise}
\end{cases}
\]

The random variables \( S \) and \( N \) are independent. The receiver observes the random variable \( X = S + N \).

(a) Suppose the receiver obtains the value \( X = x \). Find the signal estimate \( \hat{s}(x) \) that minimizes the conditional mean-squared error, \( E[(S - \hat{s}(x))^2 | X = x] \). Show all of the steps in your analysis, and display \( \hat{s}(x) \) as a plot versus \( x \). I suggest that you draw plots of \( P_S(s), P_N(n), P_{X,S}(x,s), P_X(x), P_{S|X}(s|x) \) as we did in class.

(b) How would you process the signal estimates \( \hat{s}(x) \) in order to recover the binary values \( \{-1, +1\} \) of \( S \)? In other words, what “decision rule” would you use to recover the bits from \( X \) based on \( \hat{s}(x) \)?

(c) For the decision rule that you developed in part b, what is the probability of a bit error for this system? Explain your reasoning.

(d) Write a MATLAB program to simulate this system, including the decision rule developed in part b. Compare the bit error rate (BER) in your simulation with the analytical probability of a bit error computed in part c.

Be sure to compare the simulated and analytical BER!

2. Please answer the following questions for the joint PMF \( P_{X,Y}(x,y) \) shown in the figure on page 3.

(a) Find the marginal PMFs \( P_X(x) \) and \( P_Y(y) \), and plot them.

(b) Find the mean and variance of \( X \) and \( Y \): \( \mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2 \).

(c) Find the correlation between \( X \) and \( Y \), defined as \( r_{X,Y} = E[XY] \).

(d) Find the covariance \( \text{Cov}[X,Y] \).
(e) Find the correlation coefficient \( \rho_{X,Y} \).

(f) Are \( X \) and \( Y \) independent random variables? Recall that \( X \) and \( Y \) are independent if and only if \( P_{X,Y}(x,y) = P_X(x)P_Y(y) \).

(g) Find the conditional PMFs

\[
P_{X|Y}(x|y) = P[X = x|Y = y]
\]

\[
P_{Y|X}(y|x) = P[Y = y|X = x]
\]

and plot each of these versus \( x \) and \( y \) (on separate plots).

(h) Suppose you need to produce an estimate \( \hat{x} \) of \( X \) that minimizes the mean squared error \( E[(X - \hat{x})^2] \). You must do this with no information about \( Y \). What value should you choose for \( \hat{x} \), and why? (Your answer should be a number!)

(i) Suppose that you observe that the random variable \( Y \) takes on the value \( y \) (i.e., \( Y = y \)). We would like to incorporate the knowledge that \( Y = y \) to improve our estimate of \( X \). For each possible value of \( y \), what is your estimate of \( X \), denoted by \( \hat{x}(y) \)? Explain how to compute \( \hat{x}(y) \), and present a plot of \( \hat{x}(y) \) versus \( y \).

(j) Is \( \hat{x}(y) \) in part (i) different from \( \hat{x} \) in part (h)? Is this reasonable based on the probability values in the joint PMF? Is this reasonable based on the value of correlation coefficient \( \rho_{X,Y} \) that you computed in part (e)? That is, does the value of \( \rho_{X,Y} \) lead you to expect that information about \( Y \) should be useful in predicting the value of \( X \)?

(k) Using your answer from part (h), compute \( E[(X - \hat{x})^2] \).

Using your answer from part (i), compute

\[
E[(X - \hat{x}(-1))^2|Y = -1], \ E[(X - \hat{x}(0))^2|Y = 0], \ \text{and} \ E[(X - \hat{x}(1))^2|Y = 1].
\]

Do these results show that we get better estimates for \( X \) when the value of \( Y \) is known? Please explain.