Siegel Modular Forms and Their Satake Parameters

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Why one might study Siegel modular forms

- they are multivariate elliptic modular forms
- they can be related to the number of ways of representing a quadratic form by another
- they have myriad applications: Coding Theory (Choie, Duke), Conformal Field Theory (Tuite), Special Values of $L$-functions (Fukuda-Komatsu), etc.
What’s needed for modular forms

- an upper half-space
- an arithmetic group acting on the upper half-space
- a functional equation and automorphy factor
- a Fourier expansion
An Upper-Half Space

Let

$$\mathfrak{h}_g = \{ Z \in M_g(\mathbb{C}) : Z = {}^t Z, \text{Im}(Z) > 0 \}$$

be the **Siegel upper half-space of genus** $g$.

- $\mathfrak{h}_1 =$ Poincaré upper half plane
- Since the $Z \in \mathfrak{h}_g$ are symmetric $g \times g$ matrices, we see there are $\frac{g(g+1)}{2}$ free variables.
- For $g > 1$ the upper half-space is hard to picture. In particular, $\mathfrak{H}_2$ is bounded by 28 algebraic surfaces.
An Arithmetic Group

Let \( \text{Sp}_g(\mathbb{R}) \) be

\[
\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2g}(\mathbb{R}) : \, ^t BD, \, ^t AC \text{ symm.} \, \, ^t AD - ^t CB = I_g \right\},
\]

the \textbf{symplectic group} of size \( 2g \).

- \( \text{Sp}_1(\mathbb{R}) = \text{SL}_2(\mathbb{R}) \).
- \( \Gamma_g = \text{Sp}_g(\mathbb{Z}) \) is \textbf{Siegel's modular group}. The notion of congruence subgroups of \( \Gamma_g \) also translate nicely.
An action

Let \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_g(\mathbb{R}) \). Then

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}
\]

defines an action on \( \mathfrak{h}_g \).

- For \( g = 1 \), this corresponds to the action of \( \text{SL}_2(\mathbb{R}) \) on the upper half-plane.
- We must show that \( CZ + D \) is invertible, but that’s a straightforward exercise in linear algebra.
Let $\mathcal{M}_k(\Gamma_g) = \mathcal{M}_k^g$ be the space of Siegel modular forms of weight $k$ and genus $g$. I.e., $F \in \mathcal{M}_k^g$ iff

- $F : \mathfrak{h}_g \rightarrow \mathbb{C}$ is holomorphic,
- $F \left( (AZ + B)(CZ + D)^{-1} \right) = \det(CZ + D)^k F(Z)$ for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$
- $F(Z) = \sum_{T \geq 0} a(T) e^{\pi i \text{tr}(TZ)}$ where $T$ runs over all positive semi-definite even integral $g \times g$ matrices.

We remark that

- the existence of a Fourier expansion like the one above is a theorem for $g \geq 2$
- if the Fourier expansion is supported only on positive definite forms $F \in \mathcal{S}_k^g$, i.e., is a cusp form.
Let $S$ be an even unimodular matrix of size $m$ (by a result of Hecke we know such a thing exists iff $8 \mid m$). Then for $Z \in h_g$, define

$$\Theta_S^{(g)}(Z) = \sum_{N \in M_{m \times g}} e^{\pi i \text{tr}(tNSNZ)}.$$

if $r(S, T)$ is the number of ways of representing $T$ by $S$, then

$$\Theta_S^{(g)}(Z) = \sum_{N \geq 0} r(S, N) e^{\pi i \text{tr}(TZ)}$$

$\Theta_S^{(g)}(Z) \in M^g_{m/2}$. 

An Example
Hecke operators

Let $\Gamma := \Gamma_g$ and $G = \text{GSp}_g^+(\mathbb{Q})$ be the group of rational symplectic similitudes with positive scalar factor.

Let $L(\Gamma, G)$ be the free $\mathbb{C}$-module generated by the right cosets $\Gamma \alpha$ where $\alpha \in \Gamma G$.

$\Gamma$ acts on $L(\Gamma, G)$ be right multiplication and we set

$$\mathcal{H}_g(\Gamma, G) = L(\Gamma, G)^\Gamma.$$
Hecke operators form an algebra

Let $T_1, T_2 \in \mathcal{H}_g(\Gamma, G)$ and

$$T_i = \sum_{\alpha_i \in \Gamma \backslash G} c_i(\alpha) \Gamma \alpha.$$ 

Then

$$T_1 T_2 = \sum_{\alpha, \alpha' \in \Gamma \backslash G} c_1(\alpha) c_2(\alpha') \Gamma \alpha \alpha'.$$
Local Hecke algebras

- $\mathcal{H}_g = \bigotimes_{p \text{ prime}} \mathcal{H}_{g,p}$ where the construction of the local Hecke algebra $\mathcal{H}_{g,p}$ is the same as before but with $G$ replaced with $G_p = G \cap \text{GL}_2(g)(\mathbb{Z}[p^{-1}])$.

- $\mathcal{H}_{g,p}$ is generated by the double cosets

$$T(p) = \Gamma \text{diag}(l_g, pl_g) \Gamma$$

and

$$T_i(p^2) = \Gamma \text{diag}(l_i, pl_{g-i}; p^2 l_i, pl_{g-i}) \Gamma.$$
Slash operator

$\mathcal{H}_g$ acts on $\mathcal{M}_k^g$ by

$$F|_k \left( \sum c_i \Gamma \alpha_i \right) = \sum c_i F|_k \alpha_i$$

where

$$(F|_k \alpha)(Z) = r(\alpha)^{gk - \frac{g(g+1)}{2}} \det(CZ + D)^{-k} F(\alpha \cdot Z)$$
Satake isomorphism

In the 1960s Satake proved the following theorem (in much more generality):

\[ \mathcal{H}_{g,p} \cong \mathbb{C}[x_0^{\pm 1}, \ldots, x_g^{\pm 1}]^{W_g} \]

where \( W_g \) is the Weyl group generated by the permutations of \( x_1, \ldots, x_g \) and by the maps \( x_0 \mapsto x_0 x_j, x_j \mapsto x_j^{-1}, x_i \mapsto x_j \) (\( i \neq j, 1 \leq i \leq g \)).
What Satake really proved (again in more generality) was:

\[ \text{Hom}_\mathbb{C} (\mathcal{H}_{g,p}, \mathbb{C}) = (\mathbb{C}^\times)^{g+1} / \mathcal{W}_g. \]

Let \( \Psi \) denote the isomorphism.

Let \( F \) be an eigenform for all the Hecke operators and for \( T \in \mathcal{H}_g \) write \( F|_k T = \lambda_F(T)F \). Then

\[ \Psi(T \mapsto \lambda_F(T)) = (\alpha_{0,p}, \ldots, \alpha_{g,p}). \]

The entries of the above \((g + 1)\)-tuple are the **Satake parameters** of \( F \).
Lifts

Let $f$ be an (elliptic) simultaneous eigencuspform of weight $2k$. I.e., $f \in S_{2k}^g$. Ikeda (2001) showed that (roughly) there exists a form $F \in S_{2k+g}^g$ (if $k, g$ have the same parity) so that the $L$-functions of $f$ and $F$ (almost) coincide. In particular,

$$L^{\text{std}}(F, s) = \zeta(s) \prod_{i=1}^{2g} L(f, s + k + g - i).$$

where

- $\zeta(s)$ is the zeta function
- $L(f, s) = \sum_{n \geq 1} \frac{a(n)}{n^s}$ where $a_n$ are the Fourier coefficients of $f$.
- $L^{\text{std}}(F, s) = \prod_p L_p(F, s)(p^{-s})^{-1}$ where

$$L_p(F, X) = \prod_{i=1}^{g} (1 - \alpha_{i,p} X)(1 - \alpha_{i,p}^{-1} X).$$
Why might one care about Satake parameters?

- Knowing them and knowing how to compute the Satake isomorphism lets us compute any (local) Hecke eigenvalue.
- The $L$-function(s) attached to Siegel modular forms are all defined in terms of them.
- The Ramanujan-Petersson conjecture says $|\alpha_{1,p}| = \cdots = |\alpha_{g,p}| = 1$.
- The Ikeda lift is defined in terms of the Satake parameters of the form to be lifted.
How I tackle problems for Siegel modular forms

- What’s the statement/solution to the problem for genus 1 modular forms?
- What’s the statement/solution to the problem for lifts?
- What’s the statement/solution for nonlifts?
The questions I’ve asked:

▶ (1) How can I compute the Satake parameters of a Siegel modular form?
▶ (2) What kinds of complex numbers are the Satake parameters?
Ramanujan $τ$-function

- $τ(n)$ is defined by:

$$ (2π)^{-12} Δ(z) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} τ(n)q^n $$

- the $L$-function associate to $Δ$ can be thought to have denominator at $p$ of:

$$ 1 - τ(p)X + p^{11}X^2 $$

- Ramanujan conjectured the roots of this polynomial were complex conjugate
Satake parameters of elliptic forms

Let \( f \in S_k^1 \) be a simultaneous eigenform, and let \( T(p)|_k f = \lambda_p f \). Then \((\beta_{0,p}, \beta_{1,p})\) is the solution to

\[
\beta_0^2 \beta_1 = p^{k-1} \\
\beta_0 (1 + \beta_1) = \lambda_p
\]

\( \beta_0 \beta_1 \) and \( \beta_0 \) are roots of

\[
1 - a(p)X + p^{k-1}X^2
\]

where \( a(p) \) is the \( p \)th eigenvalue of \( f \)
Satake parameters of lifts . . .

- The parameters of a lift (to genus 2) are given by:

\[
\begin{align*}
\alpha_{0,p} &= p^{k-2} \\
\alpha_{1,p} &= p^{2-k} \beta_{0,p} \\
\alpha_{2,p} &= p^{2-k} \beta_{0,p} \beta_{1,p}
\end{align*}
\]

- Similar formulas hold for arbitrary \( g \)
Sample **SAGE** code

```python
sage: p = 2
sage: k = 20
sage: g = 2
sage: M = ModularSymbols(1,k,1)
sage: S = M.cuspidal_subspace()
sage: h = S.hecke_polynomial(p)
```
Sample SAGE code

sage: R = gp.polroots(h)
sage: l2 = R[1]
sage: f = x^2-l2*x+2^(2*k-g-1)
sage: R1 = gp.polroots(f)
sage: b0 = R1[1]
sage: b1 = (l2-b0)/b0
sage: a0 = p^((4*g*k-3*g^2-2*g)/8)
sage: a1 = p^(g/2-k+1)*b0
sage: a2 = p^(g/2-k+1)*b0*b1
sage: a0,a1,a2
_16 = (262144,
  0.00086975 - 1.4142135629*I,
  0.00086975 + 1.4142135629*I)
Key result

Theorem (Andrianov)

Let $F$ be a Siegel Hecke eigenform with Satake parameters $(\alpha_{0,p}, \ldots, \alpha_{g,p})$. Then, for $T \in \mathcal{H}_{g,p}$

$$
\lambda_F(T) = \Omega(T) |_{x_i = \alpha_{i,p}}.
$$
Key result to compute with

Theorem (R. 2006)

Grade the Hecke algebra. Then, the matrix representation of $\Omega$ restricted to a direct summand is square and upper triangular. Moreover, the entries of the matrix are computed explicitly.

- It appears that this result might have some application to the study of buildings.
Satake parameters of nonlifts

In genus 2 are the roots of the following polynomial (R. (2005)):

\[ P_4(x) = x^4 - c_1 x^3 + (c_2 + 2)x^2 - c_1 x + 1 \]

where \( c_1 \) and \( c_2 \) are explicit constants that depend on \( p \) and the form \( F \).
Sample SAGE code

sage: p = 2
sage: k = 20
sage: t2 = -2^8*3^2*5*73
sage: t4 = 2^16*523*7243
sage: M22 = MatrixSpace(QQ,2,2)
sage: A = M22([1,1,2,1])
sage: V2 = VectorSpace(QQ,2)
sage: v1 = t4-p^(2*k-3)*(1/p^3+(p^2-1)/p^3+(2*p-2)/p)
sage: v2 = t2^2-4*p^(2*k-3)
sage: b = V2([v1,v2])
sage: s = A^-1 * b
Sample **SAGE** code . . .

```python
sage: l1 = (p^2-1)/p^3*p^(2*k-3)+s[0]/p
sage: l2 = ((2*p-2)/p)*p^(2*k-3)+(p-1)*s[0]/p+s[1]
sage: c1 = p*l1/p^(2*k-3)-(p^2-1)/p^3
sage: c2 = (l2+(p-1)*s[0]/p)/p^(2*k-3)-(2*p-2)/p
sage: x = PolynomialRing(RationalField()).gen()
```
Computing in Arbitrary Genus

Theorem (R.-Shemanske 2005)

Let $F \in S_k(\Gamma_g)$ be a simultaneous eigenform. There is an explicit algorithm $\mathcal{A}$ which, given the eigenvalues of $F$ with respect to the generators $T(p), T_1(p^2), \ldots, T_g(p^2)$ of the local Hecke algebra, computes the Satake $p$-parameters of $F$.

- $\mathcal{A}$ produces a polynomial of degree $2g$. For low genus ($g = 2, 3, 4$) the Satake $p$-parameters of $F$ are the roots of the polynomial. I’m working on generalizing this to all $g$. 
Implications of Algorithm

▶ The output of $A$ is a palindromic polynomial. I’ve proved a theorem that allows me to deduce if the polynomial has unimodular roots. So I can prove RPC for specific forms.

▶ The polynomials have algebraic number coefficients so the Satake parameters are themselves are algebraic numbers.

▶ This begins answering (2), but I’ve begun to look at their degrees (in genus 2).
Degrees of Satake parameters

We have the following bounds for the degrees:

- **For lifts:**
  \[
  \deg \alpha_0 = 1 \quad \deg \alpha_1 = \deg \alpha_2 = d \text{ or } 2d
  \]
  where \( d = \dim S^1_k \).

- **For nonlifts:**
  \[
  \deg \alpha_0 \leq 4(d^?)^5 \quad \deg \alpha_1 = \deg \alpha_2 \leq 4(d^?)^2
  \]
  (where \( d^? \) is the dimension of \( S^?_k(\Gamma_2) \), the space of nonlifts).
Conjectures About Satake parameters

- Computational evidence suggests that for \textit{lifts}

  \[ \deg(\alpha_0) = \deg(\alpha_1) = \deg(\alpha_2) = 2 \dim(S^1_k) \]

  while for \textit{nonlifts}

  \[ \deg(\alpha_0) = 1 \text{ and } \deg(\alpha_1) = \deg(\alpha_2) \leq \dim(S^2_k(\Gamma_2)) \]
Why we might care and what’s next

- Compare to Maeda’s conjecture
- Knowing how to distinguish lifts and nonlifts is a fundamental problem in Siegel modular forms.
  - e.g., RPC fails for lifts
- My results are limited by the number of computed examples. I plan to rectify this by developing a theory of modular symbols for genus 2 Siegel modular forms (this is how, e.g., SAGE computes Hecke eigenvalues).
A Plug (MSRI/SAGE)

- Summer Graduate Workshop in Computational Number Theory (MSRI, 7/31-8/11, 2006)
- Details on the website
  - http://www.msri.org/calendar/sgw/WorkshopInfo/392/show_sgw
- I’ll likely be leading a group of grad students looking at Siegel modular forms
- Check out the book